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Approximating Optimal Bounds in Prompt-LTL Realizability in Doubly-exponential Time*

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Abstract. In this short note, we consider the optimization variant of the realizability problem for specifications in Prompt Linear Temporal Logic (Prompt-LTL), which extends Linear Temporal Logic (LTL) by the prompt eventually operator whose scope is bounded by a parametric bound. In the realizability optimization problem, one is interested in computing the optimal bound that allows to realize a given specification. It is known that this problem is solvable in triply-exponential time, but not whether it can be done in doubly-exponential time, i.e., whether it is just as hard as solving LTL realizability. We take a step towards resolving this problem by showing that the optimum can be approximated within a factor of 2 in doubly-exponential time.

1 Introduction

The realizability problem for PROMPT–LTL, Linear Temporal Logic (LTL) enriched with eventually operators of bounded scope, should be treated as an optimization problem: determine the smallest bound on the bounded eventually such that the specification is realizable with respect to that bound. The best exact algorithms for this problem have triply-exponential running times, i.e., they are exponentially slower than algorithms for the decision variant ("does there exist a bound?"), which is 2EXPTIME-complete. We take a step towards resolving the complexity of the optimization problem by presenting an approximation algorithm with doubly-exponential running time that returns a bound that is at most twice as large as the optimum.

LTL is the most prominent logic for specifying reactive systems, but lacks the ability to express time-bounds, e.g., the formula $\mathbf{G}\,(q \rightarrow \mathbf{F}\,p)$ expresses that every request q has to be responded to by a response p. However, it does *not* require a bound on the waiting times between requests and responses, i.e., it is even satisfied if the waiting times diverge. Several parameterized logics where introduced to overcome this shortcoming [1,3,5,11]. Here, we focus on the smallest logic: PROMPT–LTL extends LTL by the prompt eventually operator $\mathbf{F_P}$, whose semantics are defined with respect to a given bound k. For example, the formula $\mathbf{G}\,(q \rightarrow \mathbf{F_P}\,p)$ is satisfied with respect to k, if every request is responded

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to within at most k steps. In decision problems for this logic the bound is quantified existentially, e.g., the model checking problem asks for a given transition system and a given formula whether there exists a bound k such that every trace of the system satisfies the specification with respect to k.

Kupferman et al. showed that PROMPT–LTL has the same desirable algorithmic properties as LTL. In particular, model checking is PSPACE-complete and realizability is 2EXPTIME-complete. Hence, one can add the prompt eventually to LTL for free. However, as already noticed by Alur et al. in their work on Parametric LTL (which also allows the dual of the prompt eventually), one can view decision problems for these parameterized logics as optimization problems: instead of asking for some bound, one should search for an optimal one. They showed that the model checking optimization problem for unipolar PLTL specifications, which includes PROMPT–LTL, can be solved in polynomial space. Thus, even finding optimal bounds is not harder than solving the LTL model checking problem. However, for realizability, or equivalently, for infinite games, the situation is different: while the decision problem is known to be 2EXPTIME-complete, the best algorithm for the optimization problem has triply-exponential running time [10].

We show that an approximately optimal bound can be determined in doubly-exponential time using the alternating color technique, which was introduced by Kupferman et al. to solve the decision problems for PROMPT–LTL. For the sake of simplicity, we present the algorithm for PROMPT–LTL, but it is applicable to parametric LTL and parametric LDL as well.

2 Definitions

Throughout the paper, we fix a finite set P of atomic propositions. The non-negative integers are denoted by $\mathbb{N}$.

2.1 Prompt-LTL

The formulas of PROMPT–LTL are given by the grammar

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{X}\,\varphi \mid \varphi\,\mathbf{U}\,\varphi \mid \varphi\,\mathbf{R}\,\varphi \mid \mathbf{F_P}\,\varphi,$$

where $p \in P$. We use $\varphi \rightarrow \psi$ as shorthand for $\neg\varphi \vee \psi$, where we require φ to be a (negated) atomic proposition and identify $\neg\neg p$ with p.

The set of subformulas of a PROMPT–LTL formula φ is denoted by $\mathrm{cl}(\varphi)$ and we define the size $|\varphi|$ of φ to be the cardinality of $\mathrm{cl}(\varphi)$.

In order to evaluate PROMPT–LTL formulas, we need to fix a bound $k \in \mathbb{N}$ to evaluate the prompt eventually operator. The satisfaction relation is defined for an ω-word $w \in \left(2^P\right)^\omega$, a position n of w, a bound k, and a PROMPT–LTL formula. The definition is standard for the classical operators and defined as follows for the prompt eventually:

- $(w, n, k) \models \mathbf{F_P}\,\varphi$ if and only if there exists a j in the range $0 \leq j \leq k$ such that $(w, n + j, k) \models \varphi$.

For the sake of brevity, we write $(w, k) \models \varphi$ instead of $(w, 0, k) \models \varphi$ and say that w is a model of φ with respect to k. Note that φ is an LTL formula [6], if it does not contain the prompt eventually. In this case, we write $w \models \varphi$.

2.2 Prompt-LTL Realizability

Throughout this subsection, we fix a partition (I, O) of P. An instance of the PROMPT–LTL realizability problem over (I, O) consists of an PROMPT–LTL formula φ over $P = I \cup O$ and asks to determine the winner in the following game, played between Player I and Player O in rounds $n = 0, 1, 2, \ldots$: in round n, Player I picks $i_n \subseteq I$ and afterwards Player O picks $o_n \subseteq O$. The resulting play is $\rho = (i_0 \cup o_0)(i_1 \cup o_1)(i_2 \cup o_2) \cdots \in (2^P)^\omega$.

A strategy for Player O is a mapping $\sigma \colon (2^I)^+ \to 2^O$. A play ρ as above is consistent with σ, if $o_n = \sigma(i_0 \cdots i_n)$ for every n. We say that σ is realizes φ with respect to $k \in \mathbb{N}$, if every play that is consistent with σ satisfies φ with respect to k. Formally, the PROMPT–LTL realizability problem asks, given a PROMPT–LTL formula φ, whether there is a strategy σ and a k such that σ realizes φ with respect to k and, if yes, to compute such a strategy. In this case, we say φ is realizable.

The LTL realizability problem is defined by restricting the specifications φ to LTL formulas and is 2EXPTIME-complete [7]. It turns out that PROMPT–LTL realizability is not harder.

Theorem 1 ([5]). *The* PROMPT–LTL *realizability problem is* 2EXPTIME-*complete. Furthermore, if φ is realizable with respect to some k, then also with respect to some k that is doubly-exponential in $|\varphi|$.*

Furthermore, the doubly-exponential upper bound on the necessary k to realize φ is tight [10]. Also, if φ is realizable with respect to some k, then also with respect to every larger k'.

2.3 The Alternating-color Technique

Our algorithm presented in the next section is based on an application of Kupferman et al.'s alternating-color technique to PROMPT–LTL realizability. We recall the technique in this subsection.

Let $p \notin P$ be a fixed fresh proposition. An ω-word $w' \in \left(2^{P \cup \{p\}}\right)^\omega$ is a p-coloring of $w \in \left(2^P\right)^\omega$ if $w'_n \cap P = w_n$, i.e., w_n and w'_n coincide on all propositions in P. We say that a position is a change point, if $n = 0$ or if the truth value of p at positions $n - 1$ and n differs. A p-block is an infix $w'_m \cdots w'_n$ of w' such that m and $n + 1$ are adjacent change points. Let $k \geq 1$: we say that w' is k-spaced, if the truth value of p changes infinitely often and each p-block has length at least k; we say that w' is k-bounded, if each p-block has length at most k (which implies that the truth value of p changes infinitely often).

Given a PROMPT–LTL formula φ, let $\mathrm{rel}(\varphi)$ denote the formula obtained by inductively replacing every subformula $\mathbf{F_P}\,\psi$ by

$$(p \to (p\,\mathbf{U}\,(\neg p\,\mathbf{U}\,\mathrm{rel}(\psi)))) \wedge (\neg p \to (\neg p\,\mathbf{U}\,(p\,\mathbf{U}\,\mathrm{rel}(\psi)))).$$

Intuitively, instead of requiring ψ to be satisfied within a bounded number of steps, $\mathrm{rel}(\varphi)$ requires it to be satisfied within at most one change point. The relativization $\mathrm{rel}(\varphi)$ is an LTL formula of size $\mathcal{O}(|\varphi|)$. Kupferman et al. showed that φ and $\mathrm{rel}(\varphi)$ are "equivalent" on ω-words which are bounded and spaced.

Lemma 1 ([5]). *Let φ be a* PROMPT–LTL *formula.*

1. *If $(w, k) \models \varphi$, then $w' \models \mathrm{rel}(\varphi)$ for every k-spaced p-coloring w' of w.*
2. *Let $k \in \mathbb{N}$. If w' is a k-bounded p-coloring of w such that $w' \models \mathrm{rel}(\varphi)$, then $(w, 2k) \models \varphi$.*

3 Approximating Optimal Bounds in Prompt-LTL Realizability

Determining whether a PROMPT–LTL formula φ is realizable with respect to some k induces a natural optimization problem: determine the smallest such k. The optimum (and a strategy realizing φ with respect to the optimum) can be computed in triply exponential time [10].

However, it is an open problem whether the optimization problem can be solved in doubly-exponential time, i.e., whether optimal PROMPT–LTL realizability is no harder than LTL realizability. We take a step towards resolving the problem by showing that the optimum can be approximated within a factor of 2 in doubly-exponential time.

The alternating-color technique is applied to PROMPT–LTL realizability problems by replacing φ by its relativization $\mathrm{rel}(\varphi)$ and by letting Player 0 determine the truth value of the distinguished proposition p for every position by adding it to the propositions in O. The full details are explained in [5], where the following statements are shown to prove the application of the alternating color technique to be correct. Here, ψ_k is an LTL formula of linear size in k that characterizes k-boundedness, i.e., $w' \models \psi_k$ if, and only if, w' is a k-bounded p-coloring.

Lemma 2 ([5]). *Let φ be a* PROMPT–LTL *formula and let $k \in \mathbb{N}$.*

1. *A strategy realizing φ with respect to k can be turned into a strategy realizing $\mathrm{rel}(\varphi) \wedge \psi_k$.*
2. *A strategy realizing $\mathrm{rel}(\varphi) \wedge \psi_k$ can be turned into a strategy realizing φ with respect to $2k$.*

Furthermore, if k is not too large, we can check the realizability of $\mathrm{rel}(\varphi) \wedge \psi_k$ in doubly-exponential time.

Lemma 3. *The following problem is in* 2EXPTIME*: Given a* PROMPT–LTL *formula φ and a natural number k that is at most doubly-exponential in $|\varphi|$, is $\mathrm{rel}(\varphi) \wedge \psi_k$ realizable? Furthermore, one can compute a realizing the formula strategy (if one exists) in doubly-exponential time.*

Proof. We reduce the problem to a parity game (see [4] for background): First, we construct a deterministic parity automaton recognizing the language

$$\{\rho \in (2^{P \cup \{p\}})^\omega \mid \rho \models \mathrm{rel}(\varphi)\}$$

and intersect it with a deterministic safety automaton that recognizes

$$\{\rho \in (2^{P \cup \{p\}})^\omega \mid \rho \models \psi_k\}.$$

It is known that the first automaton is of doubly-exponential size and has exponentially many colors (both in $|\varphi|$) while the second one is of linear size in k. Thus, the resulting deterministic parity automaton $\mathfrak{A}$ recognizing the intersection is of doubly-exponential size and linear size in k and has exponentially many colors.

Next, we split a transition of $\mathfrak{A}$ labeled by $A \subseteq P \cup \{p\}$ into two, the first one labeled by $A \cap I$ and the second one by $A \setminus I$. By declaring the original states of $\mathfrak{A}$ to be Player I states and the new intermediate states obtained by splitting the transitions to be Player O states, we obtain a parity game that is won by Player O from the initial state of $\mathfrak{A}$ if, and only if, $\mathrm{rel}(\varphi) \wedge \psi_k$ is realizable. Additionally, a winning strategy for Player O in the parity game can be turned into a strategy realizing $\mathrm{rel}(\varphi) \wedge \psi_k$.

This parity game is of doubly-exponential size with exponentially many colors, both in $|\varphi|$. The winner and a winning strategy for her in such a game can be computed in doubly-exponential time [8]. □

Relying on Lemma 2 and Theorem 1 we give an approximation algorithm for finding optimal bounds for PROMPT–LTL realizability. Given an input φ, the algorithm first checks whether φ is realizable with respect to some φ. If not, then the optimum is ∞ by convention. Otherwise, we obtain a doubly-exponential upper bound u on the optimum. Now, the algorithm determines the smallest $k \leq u$ such that $\mathrm{rel}(\varphi) \wedge \psi_k)$ is realizable and returns $2k$. The emptiness test and determining the realizability of $\mathrm{rel}(\varphi) \wedge \psi_k)$ can be executed in doubly-exponential time as argued above. As the latter problem has to be solved at most doubly-exponentially often[1], the overall running time is doubly-exponential as well. Furthermore, due to Lemma 2.2, we even obtain a strategy realizing φ with respect to $2k$.

It remains to argue that the algorithm approximates the optimum $k_{\mathrm{opt}} \leq u$ within a factor of 2: let $2k$ be the output of the approximation algorithm, i.e., k is minimal with the property that $\mathrm{rel}(\varphi) \wedge \psi_k)$ is realizable. Thus, Lemma 2.2 implies $k_{\mathrm{opt}} \leq 2k$. Conversely, φ being realizable with respect to k_{opt} implies that $(\mathrm{rel}(\varphi) \wedge \psi_{k_{\mathrm{opt}}}$ is realizable due to Lemma 2.1, i.e., $k \leq k_{\mathrm{opt}}$ due to minimality of k.

Altogether, we obtain $k \leq k_{\mathrm{opt}} \leq 2k$. Recall that the algorithm returns $2k$, i.e., φ is realizable with respect to the returned value due to monotonicity.

[1] With binary search, this can be improved to exponentially often. However, the running time of the realizability check depends on k, which is typically small.

Furthermore, the approximation ratio $\frac{2k}{2k-k_{\text{opt}}}$ is bounded by

$$\frac{2k}{2k-k_{\text{opt}}} \leq \frac{2k}{2k-k} = 2.$$

Theorem 2. *The optimization problem for* PROMPT–LTL *realizability can be approximated within a factor of* 2 *in doubly-exponential time. As a byproduct, one obtains a strategy witnessing the approximatively optimal bound.*

4 Beyond Prompt-LTL

In this section, we argue that the approximation algorithm presented in the previous section can be extended to more expressive parametric temporal logics, e.g., parametric LTL (PLTL) and parametric LDL (PLDL).

PLTL [1] extends PROMPT–LTL by allowing multiple bounds on the prompt eventually and by adding the dual operator, the parameterized always. Formally, one has operators $\mathbf{F}_{\leq z}$ and $\mathbf{G}_{\leq z}$, where z is some variable. Semantics are defined with respect to a variable valuation mapping the variables to natural numbers. The formula $\mathbf{F}_{\leq z}\varphi$ is satisfied, if φ holds within the next $\alpha(z)$ steps. Dually, the formula $\mathbf{G}_{\leq z}\varphi$ is satisfied, if φ holds for at least the next $\alpha(z)$ steps. PLTL model checking is PSPACE-complete [1] while PLTL realizability is 2EXPTIME-complete [10][2], i.e., both problems are not harder than their counterparts for LTL and PROMPT–LTL.

The realizability optimization problems are only considered for the unipolar fragment of PLTL, i.e., for formulas that only contain parameterized eventually operators ($\text{PLTL}_{\mathbf{F}}$ formulas) or only parameterized always operators ($\text{PLTL}_{\mathbf{G}}$ formulas), but not both. For $\text{PLTL}_{\mathbf{F}}$ formulas, one is interested in minimizing the minimal or maximal parameter value. Dually, for $\text{PLTL}_{\mathbf{G}}$ formulas, one is interested in maximizing the minimal or maximal parameter value. These problems are solvable in triply-exponential time [10], which is shown by a reduction to the PROMPT–LTL optimization problem that preserves the exact optimum. Hence, the PLTL optimization problems can be approximated in doubly-exponential time as well.

Theorem 3. *The optimization problems for unipolar* PLTL *realizability can be approximated within a factor of* 2 *in doubly-exponential time. As a byproduct, one obtains a strategy witnessing the approximatively optimal bound.*

The logic PLDL [3] is the parameterized extension of Linear Dynamic Logic (LDL) [2,9]. The latter replaces the temporal operators of LTL by an eventually and an always operator that are guarded by regular expressions: $\langle r \rangle \varphi$ holds, if there is a prefix that matches the regular expression r such that the corresponding suffix satisfies φ. Dually, $[r]\,\varphi$ holds, if for every prefix that matches r,

[2] The results there are presented for infinite games in finite arenas. However, they are easily transferable to the realizability setting.

the corresponding suffix satisfies φ. LDL captures the ω-regular languages while retaining the desirable algorithmic properties of LTL [2,9]

As expected, PLDL allows to equip these operators with variables that bound the length of the prefixes under consideration. All decision problems for PLDL are not harder than their counterparts for PLTL [3]. In particular, the alternating color technique can be lifted to PLDL and analogues of Lemma 2 and Theorem 1 hold as well. Hence, the approximation algorithm presented above is applicable to the optimization problems for unipolar PLDL realizability, which are defined as expected.

Theorem 4. *The optimization problems for unipolar PLDL realizability can be approximated within a factor of* 2 *in doubly-exponential time. As a byproduct, one obtains a strategy witnessing the approximatively optimal bound.*

References

1. Alur, R., Etessami, K., La Torre, S., Peled, D.: Parametric temporal logic for "model measuring". ACM Trans. Comput. Log. 2(3), 388–407 (2001)
2. De Giacomo, G., Vardi, M.Y.: Linear temporal logic and linear dynamic logic on finite traces. In: Rossi, F. (ed.) IJCAI 2013. IJCAI/AAAI (2013)
3. Faymonville, P., Zimmermann, M.: Parametric linear dynamic logic. In: Peron, A., Piazza, C. (eds.) GandALF. EPTCS, vol. 161, pp. 60–73 (2014)
4. Grädel, E., Thomas, W., Wilke, T. (eds.): Automata, Logics, and Infinite Games: A Guide to Current Research, LNCS, vol. 2500. Springer (2002)
5. Kupferman, O., Piterman, N., Vardi, M.Y.: From liveness to promptness. Formal Methods in System Design 34(2), 83–103 (2009)
6. Pnueli, A.: The temporal logic of programs. In: FOCS 1977. pp. 46–57. IEEE Computer Society (1977)
7. Pnueli, A., Rosner, R.: On the synthesis of an asynchronous reactive module. In: Ausiello, G., Dezani-Ciancaglini, M., Rocca, S.R.D. (eds.) ICALP 1989. LNCS, vol. 372, pp. 652–671. Springer (1989)
8. Schewe, S.: Solving parity games in big steps. In: Arvind, V., Prasad, S. (eds.) FSTTCS 2007. LNCS, vol. 4855, pp. 449–460. Springer (2007)
9. Vardi, M.Y.: The rise and fall of LTL. In: D'Agostino, G., La Torre, S. (eds.) GandALF 2011. EPTCS, vol. 54 (2011)
10. Zimmermann, M.: Optimal Bounds in Parametric LTL Games. Theoret. Comput. Sci. 493(0), 30 – 45 (2013)
11. Zimmermann, M.: Parameterized linear temporal logics meet costs: Still not costlier than LTL. In: Esparza, J., Tronci, E. (eds.) GandALF 2015. EPTCS, vol. 193, pp. 144–157 (2015)