Analysis of Arithmetic Prolog Programs using Abstract Interpretation

Alexander Weinert

Master Thesis Presentation

June 22, 2015

Roadmap

Program

YES/NO/
MAYBE

Roadmap

Program $\longrightarrow$ YES/NO/MAYBE

Roadmap

Program $\longrightarrow$ YES/NO/MAYBE

Question: Do all evaluations of the program terminate?

```
fac(X)  =
```

Prolog - Introduction

```
fac(X)  =
        if X > 0 then Y1 = fac(X-1), return Y1 * X
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fac(X, Y)  :-
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fac(X, Y)  :- X > 0, fac(X - 1, Y1),
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fac(X, Y)  :- X > 0, fac(X - 1, Y1), Y is Y1 * X.
fac(X, Y)  :- X =:= 0,
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$\boxed{fac(1, Res)}$

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$$\boxed{\mathit{fac}(1, \mathit{Res})}$$

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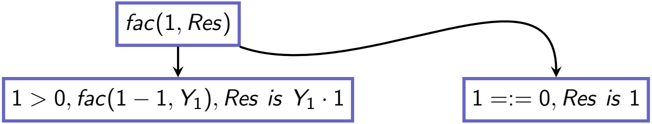

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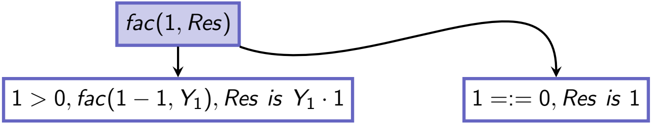

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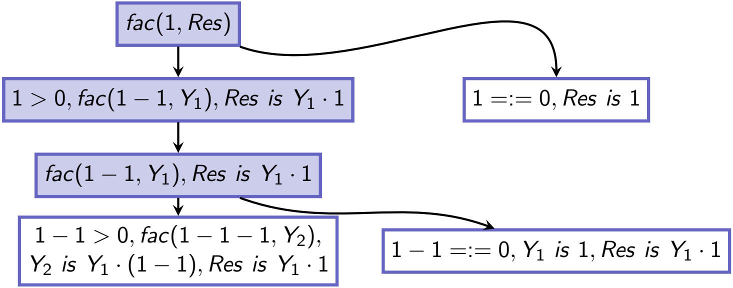

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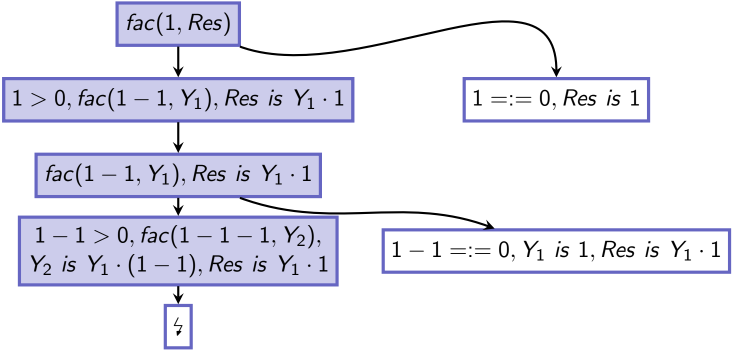

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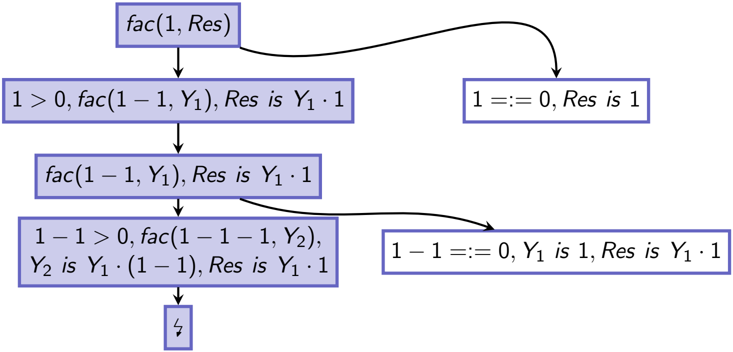

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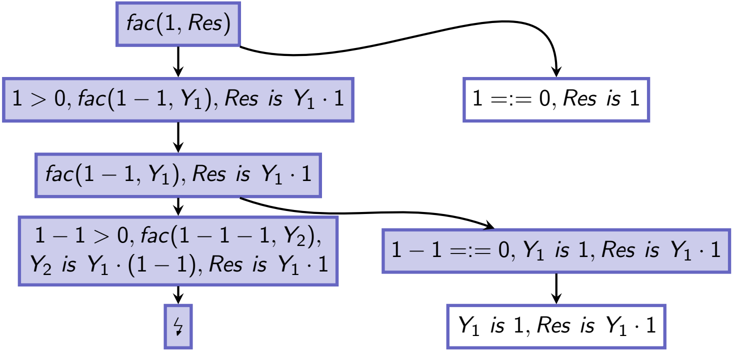

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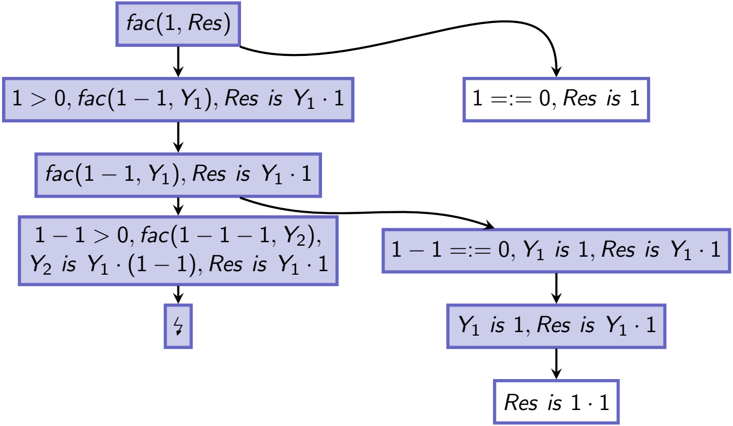

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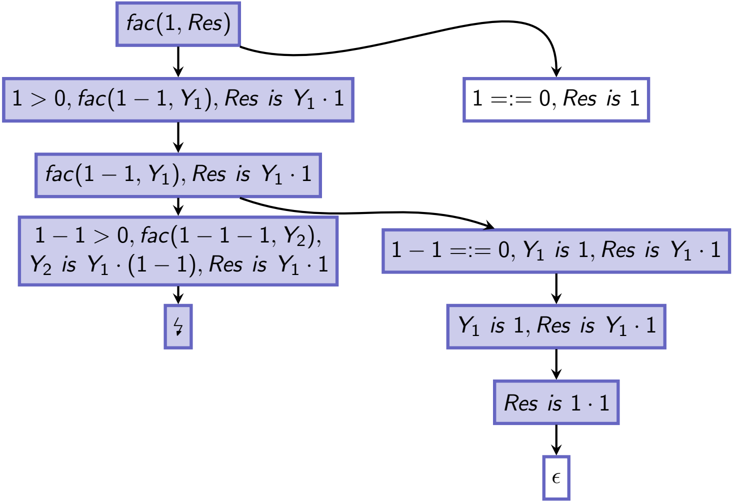

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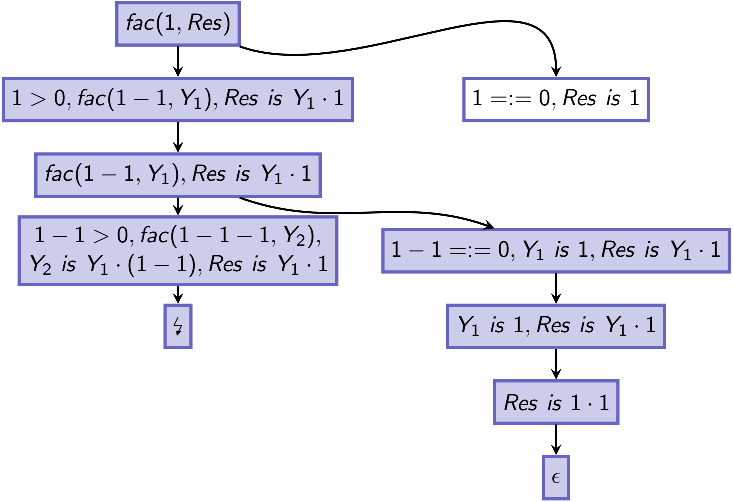

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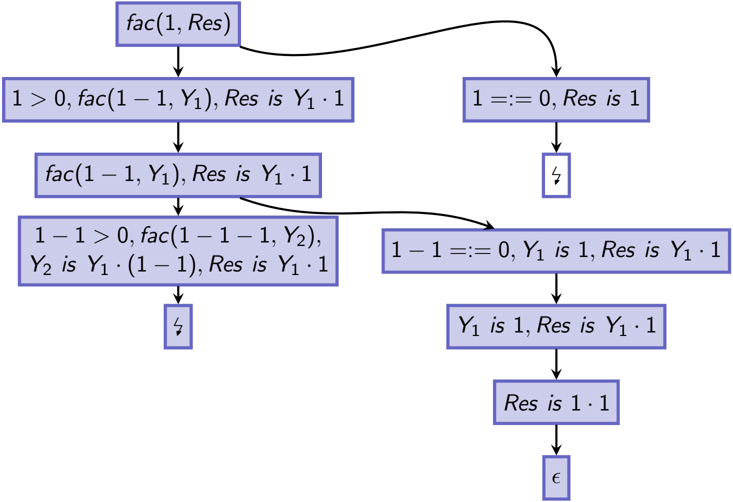

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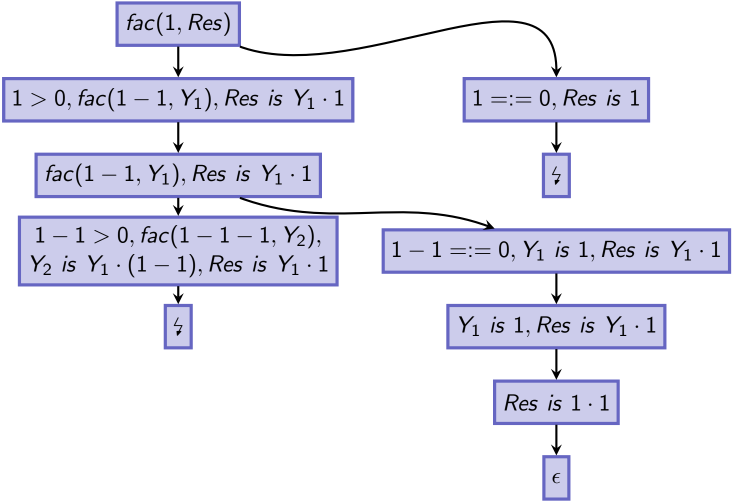

Prolog - Cut

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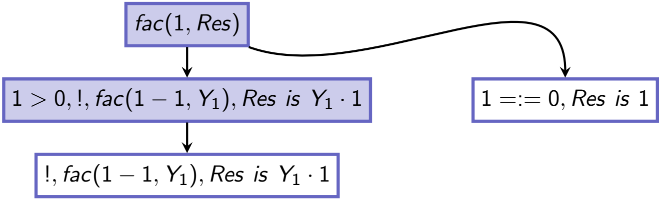
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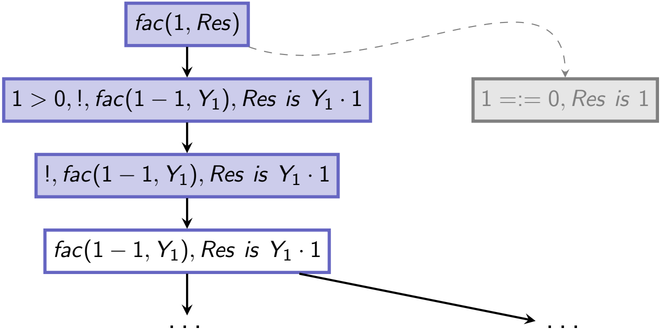
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...
...

Termination

Given: Prolog Program, some query template

Question:

Termination

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Question: For all queries matching the template:

Termination

Given: Prolog Program, some query template

Question: For all queries matching the template: Does the inference of the query on the program eventually terminate?

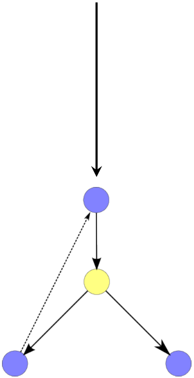
Roadmap

Program

YES/NO/
MAYBE

Roadmap

Program



Termination Graph

YES/NO/
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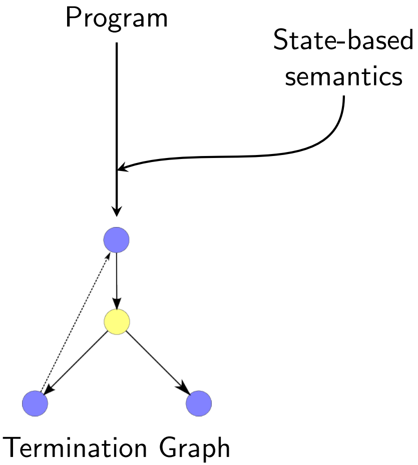
Roadmap



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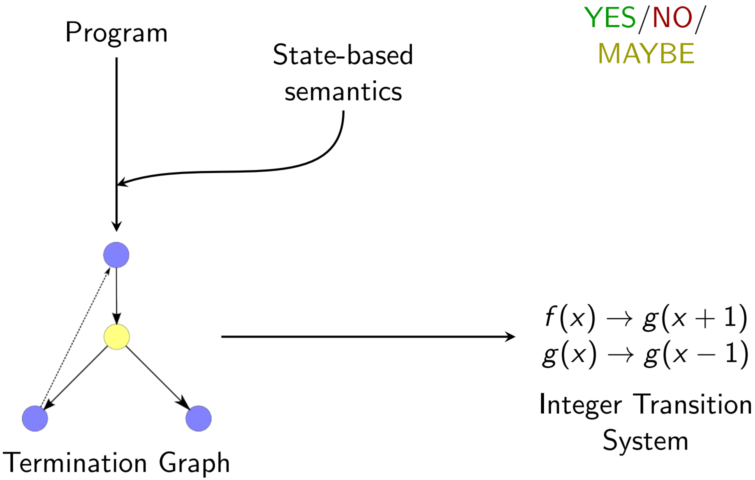
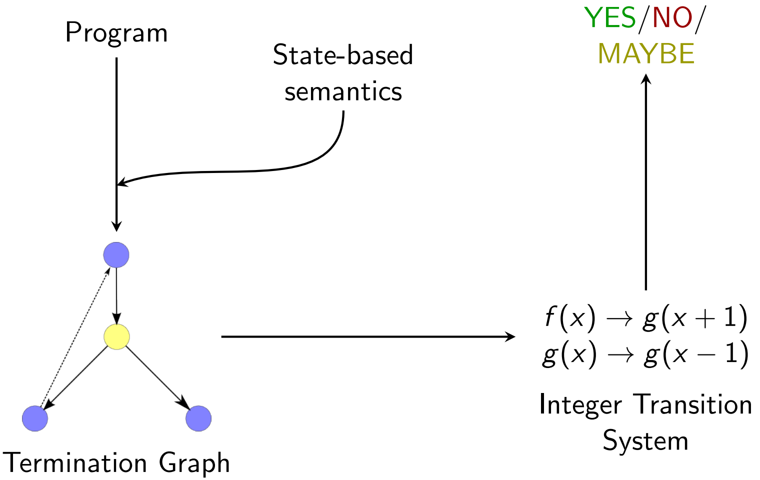


Program
State-based semantics
YES/NO/MAYBE
Termination Graph
$f(x) \to g(x+1)$
$g(x) \to g(x-1)$
Integer Transition System

Roadmap



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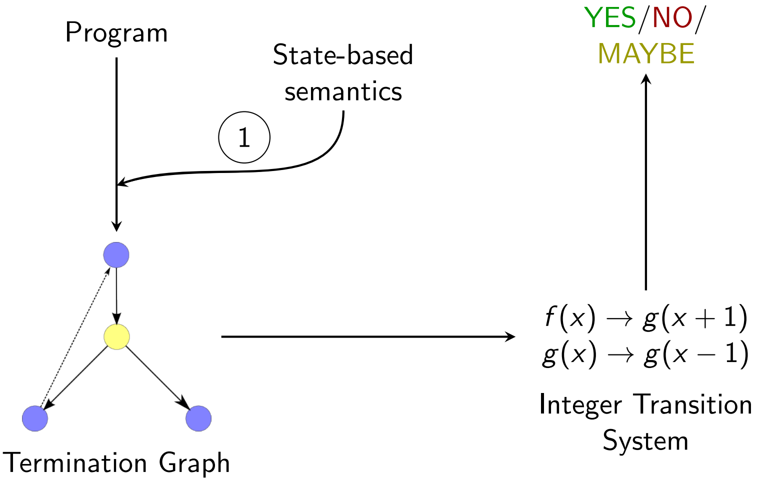

Program
State-based semantics
1
YES/NO/MAYBE
$f(x) \to g(x+1)$
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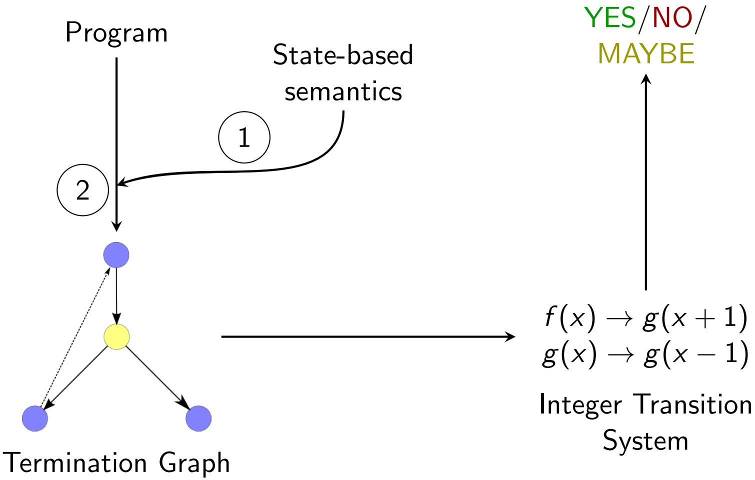
Program
State-based
semantics
1
2
YES/NO/
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Integer Transition
System
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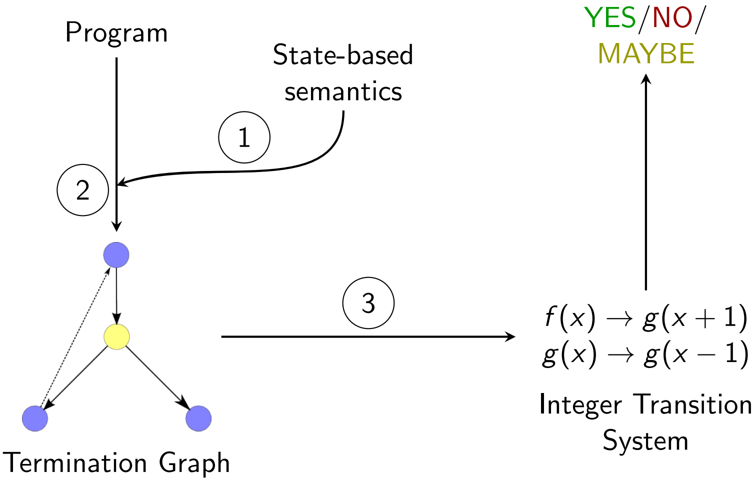
Roadmap


YES/NO/
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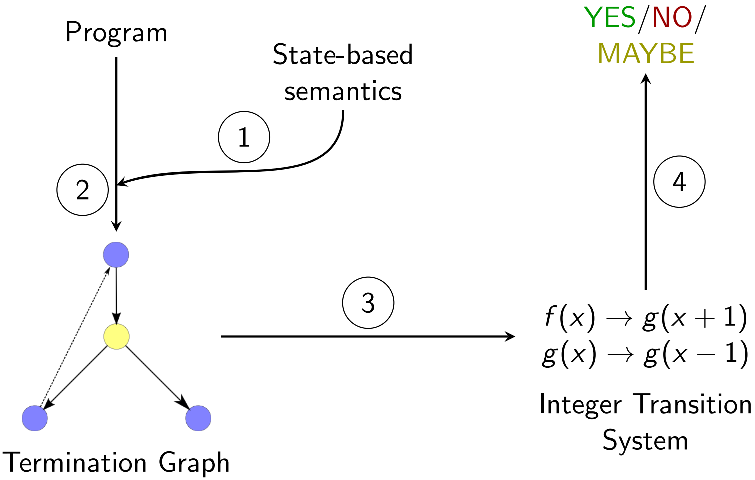

YES/NO/
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Program
State-based
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1
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4
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From Trees to States

- Prolog: Tree-based semantics
- Well-known techniques: State-based semantics

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Solution: State-based semantics for Prolog
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Solution: State-based semantics for Prolog
(Linear Operational Semantics, Ströder et al., 2012)

- Basic Idea: Leaves of tree describe state of inference
- Use front of tree as state

From Trees to States

$$\begin{tikzcd}
\boxed{\mathit{fac}(1, \mathit{Res})} \arrow[d] \arrow[rr, bend left, shift left] & & \\
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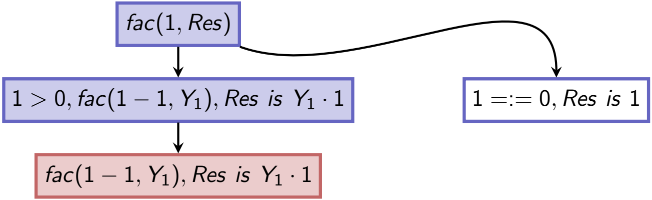
From Trees to States



$$\boxed{fac(1-1, Y1), Res\ is\ Y_1 \cdot 1}$$

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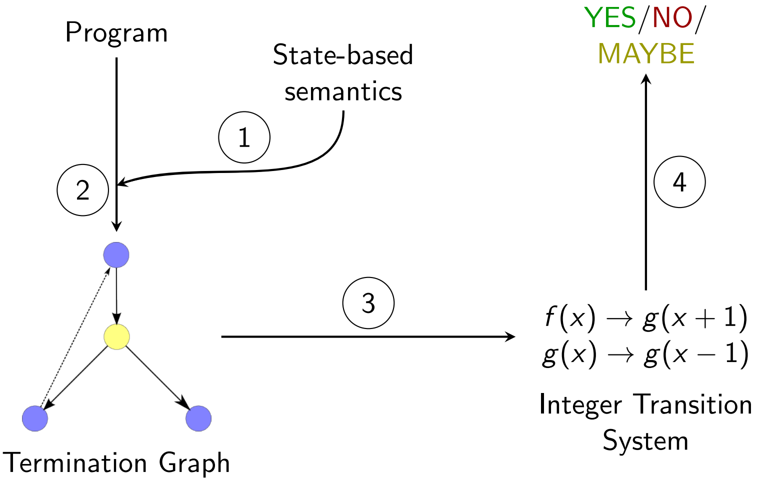
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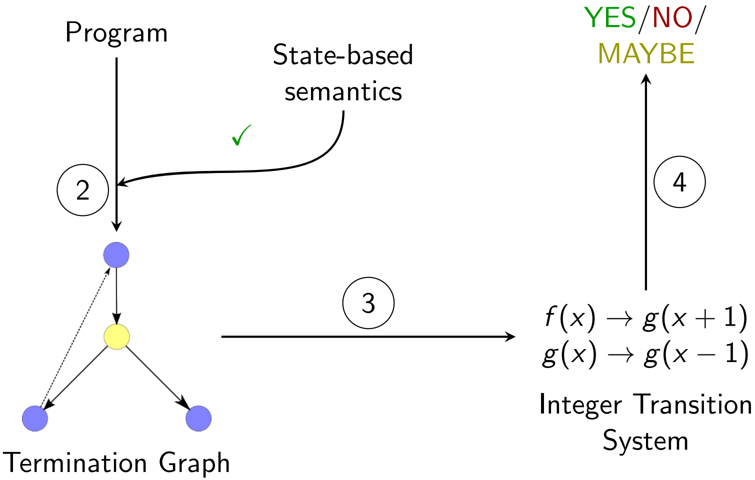

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semantics
✓
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Given: Some Program, some query template

Goal: Finite representation of all possible inferences

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Idea: Represent set of runs as graph

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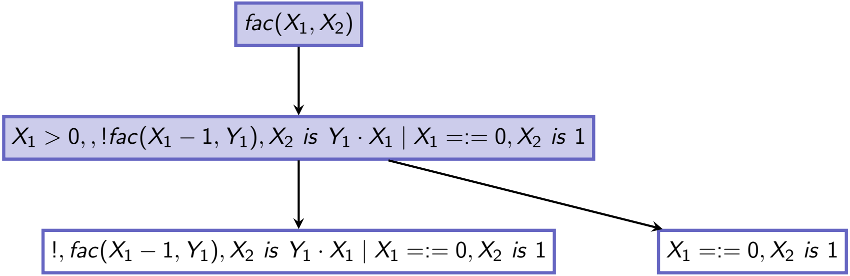

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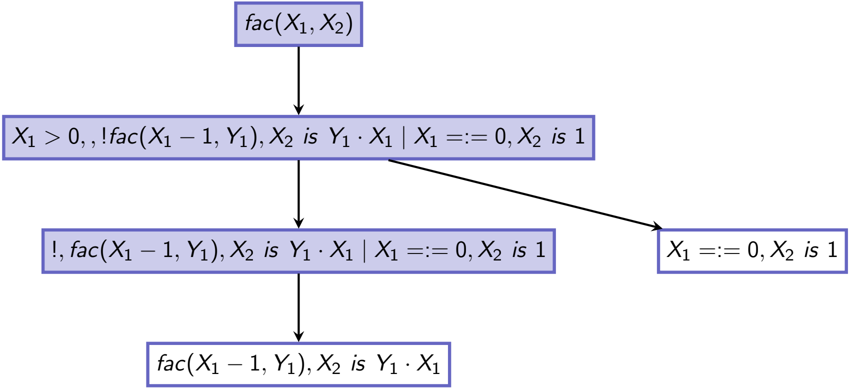
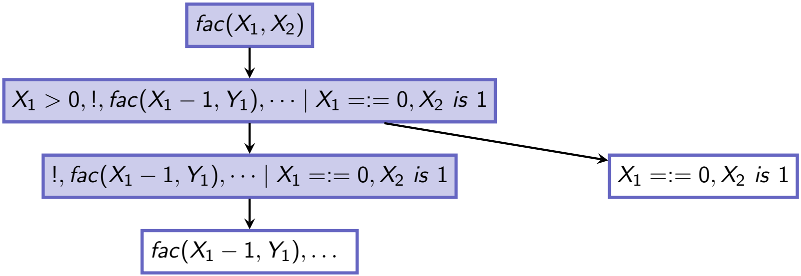
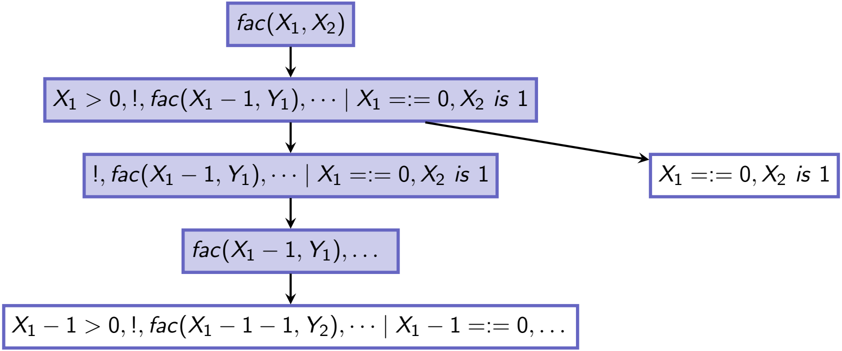


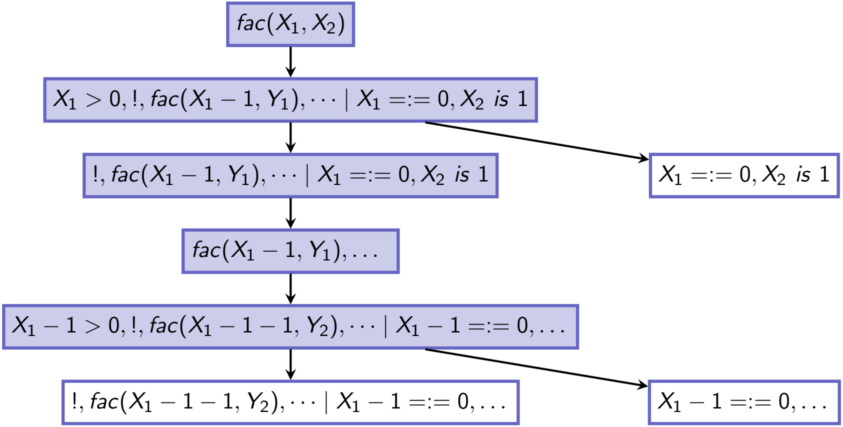
From Programs To Graphs - Nonterminating Construction



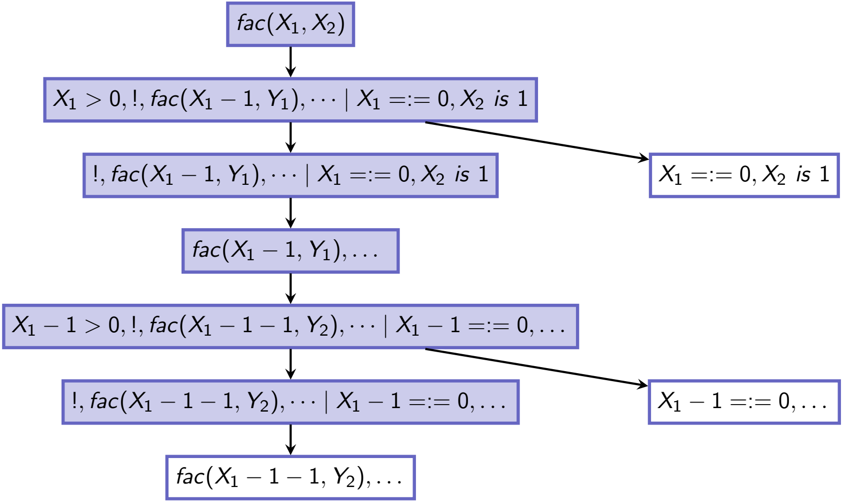
From Programs To Graphs - Nonterminating Construction



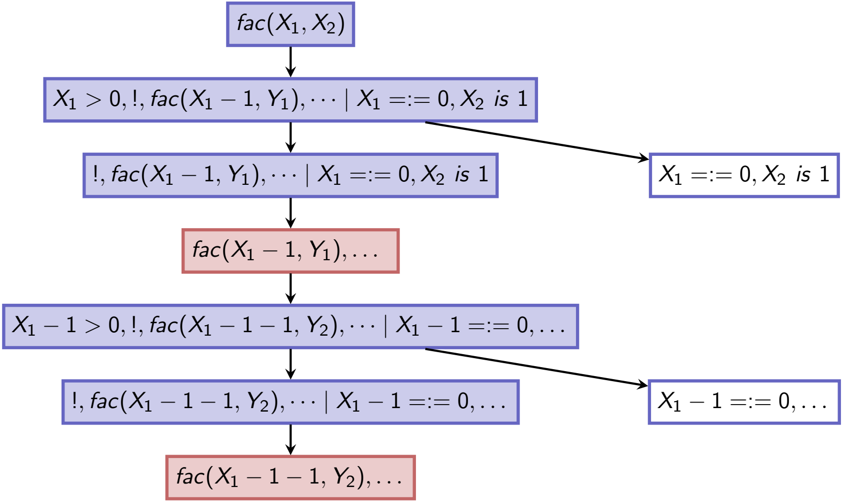
From Programs To Graphs - Nonterminating Construction



From Programs To Graphs - Nonterminating Construction



From Programs To Graphs - Nonterminating Construction



From Programs To Graphs - Split rule

$$\boxed{fac(X_1 - 1, Y_1) \quad , \quad X_2 \ is \ Y_1 \cdot X_1}$$

From Programs To Graphs - Split rule

$$\boxed{\boxed{fac(X_1 - 1, Y_1)}\ ,\quad X_2\ is\ Y_1 \cdot X_1}$$

$$\boxed{fac(X_1 - 1, Y_1)}$$

From Programs To Graphs - Split rule

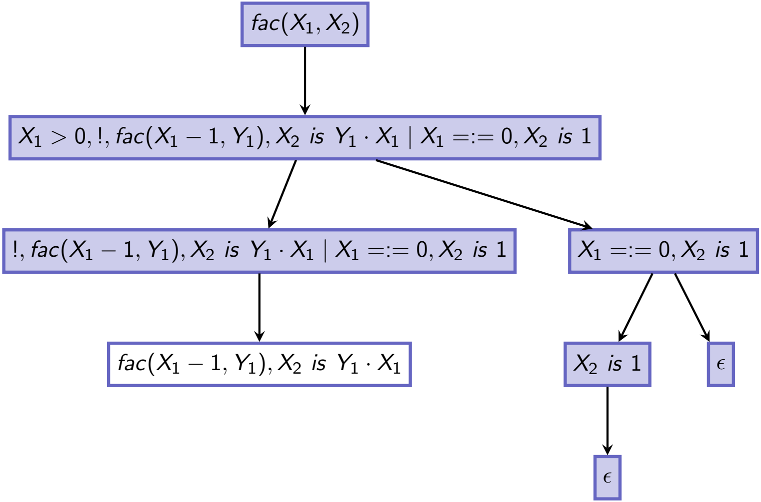
$\boxed{fac(X_1 - 1, Y_1) \;\; , \;\; \boxed{X_2 \; is \; Y_1 \cdot X_1}}$

$\boxed{fac(X_1 - 1, Y_1)}$ $\boxed{X_2 \; is \; Y_1 \cdot X_1}$

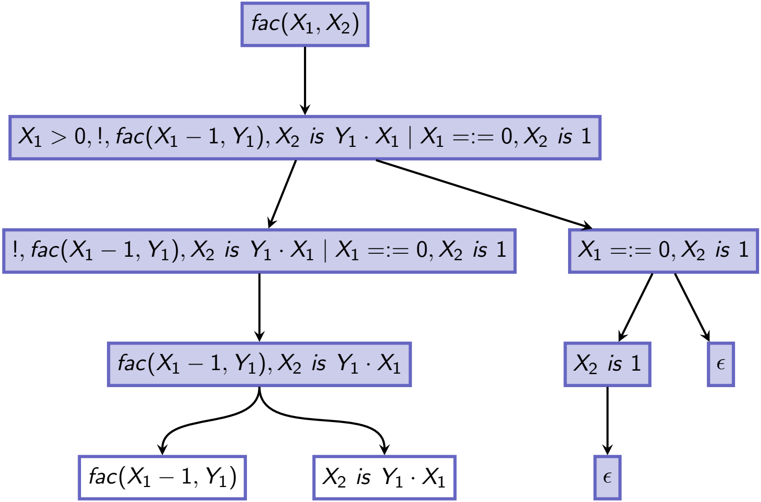
From Programs To Graphs - Split rule

$$\begin{tikzcd} & \boxed{fac(X_1 - 1, Y_1) \quad , \quad X_2 \ is \ Y_1 \cdot X_1} \arrow[dl] \arrow[dr] & \\ \boxed{fac(X_1 - 1, Y_1)} & & \boxed{X_2 \ is \ Y_1 \cdot X_1} \end{tikzcd}$$

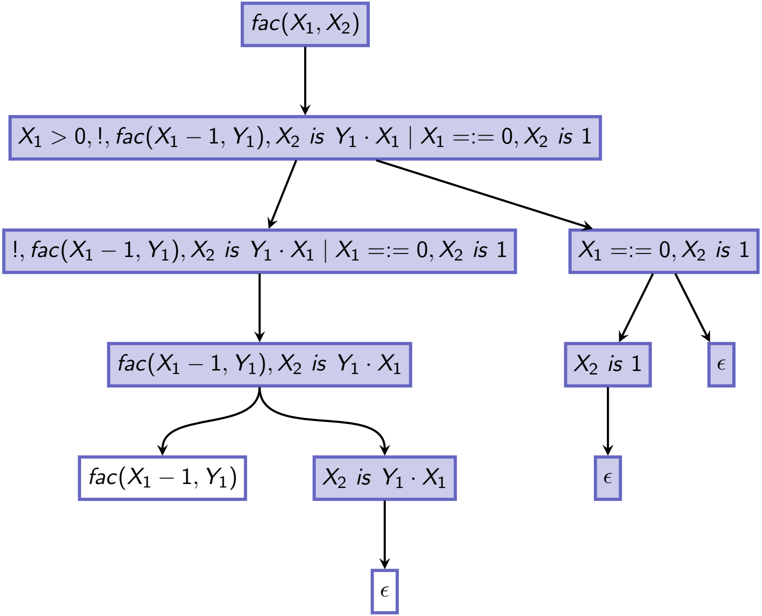
From Programs To Graphs - Split rule



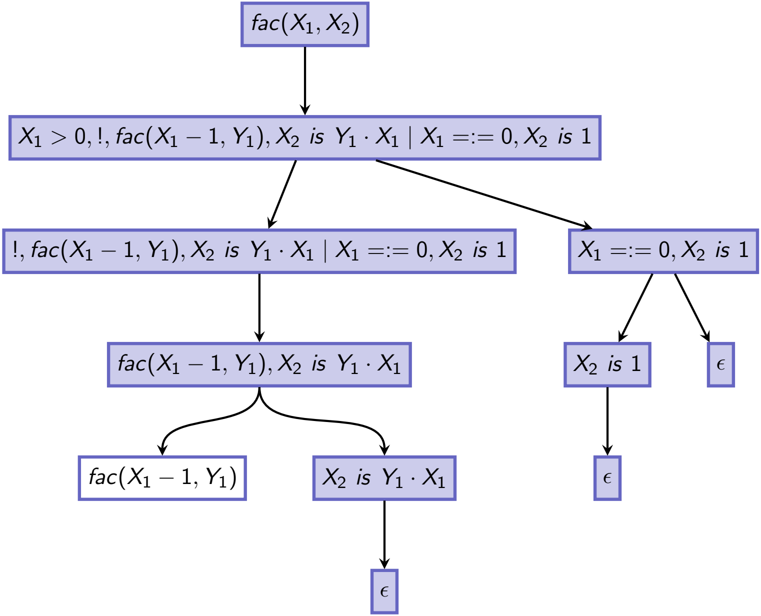
From Programs To Graphs - Split rule



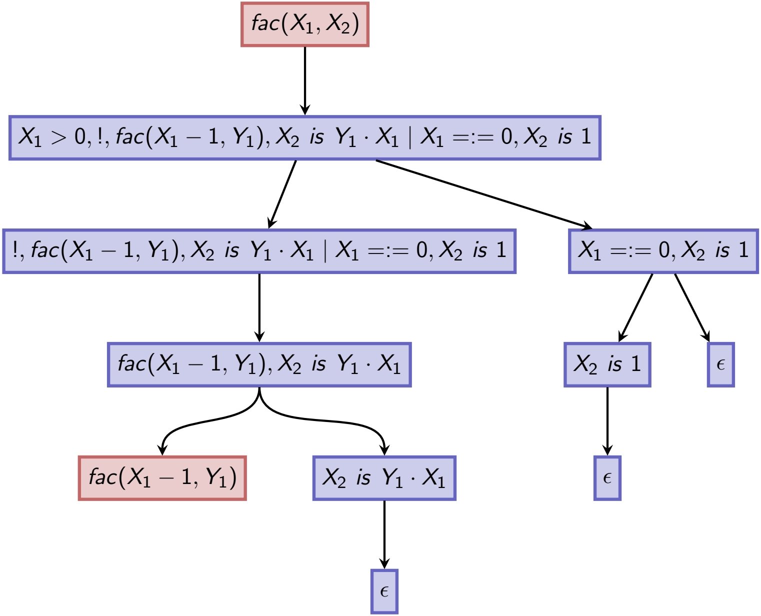
From Programs To Graphs - Split rule



From Programs To Graphs - Split rule



From Programs To Graphs - Split rule



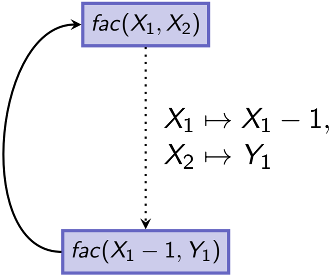
From Programs To Graphs - Instance Rule

$$\begin{tikzcd} \boxed{fac(X_1, X_2)} \arrow[d, dotted] \\ \boxed{fac(X_1 - 1, Y_1)} \end{tikzcd}$$

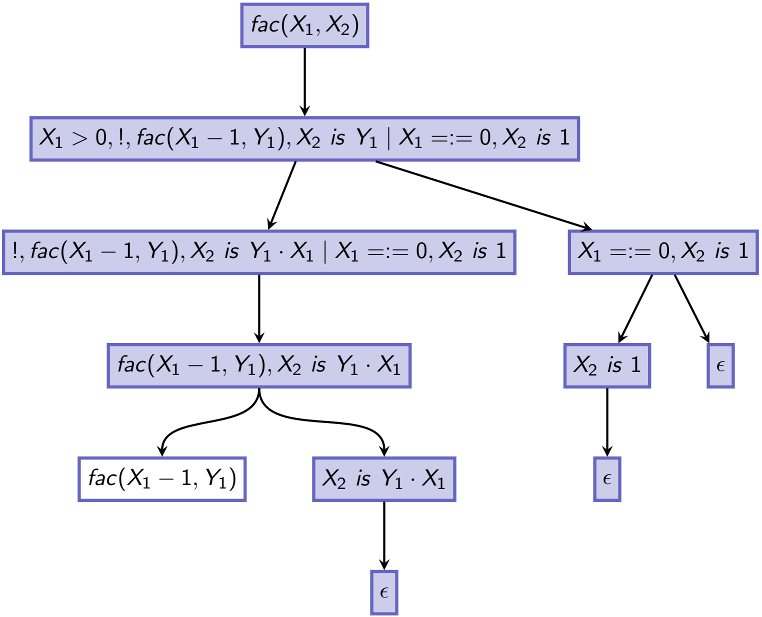
From Programs To Graphs - Instance Rule

$$\begin{tikzcd} \boxed{fac(X_1, X_2)} \arrow[d, dotted, "{\begin{array}{l} X_1 \mapsto X_1 - 1, \\ X_2 \mapsto Y_1 \end{array}}"] \\ \boxed{fac(X_1 - 1, Y_1)} \end{tikzcd}$$

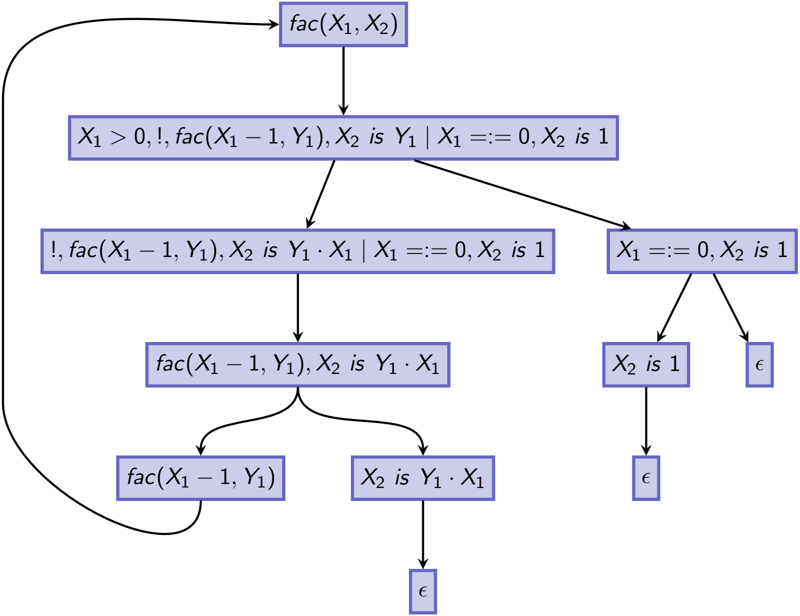
From Programs To Graphs - Instance Rule



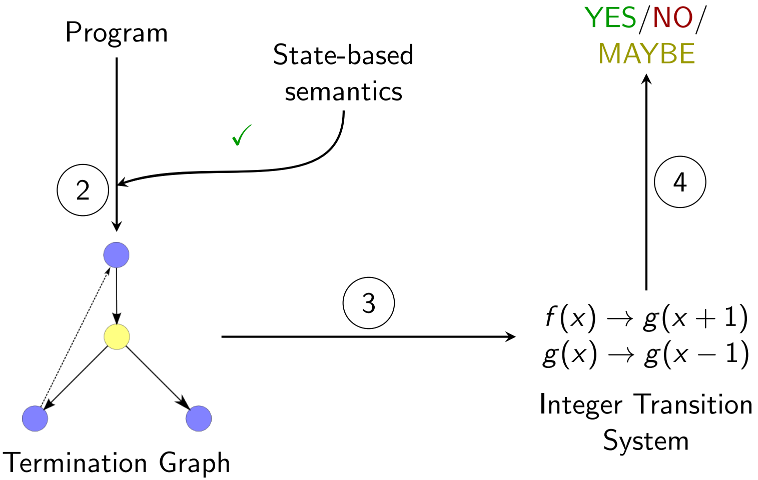
From Programs To Graphs - Final Result



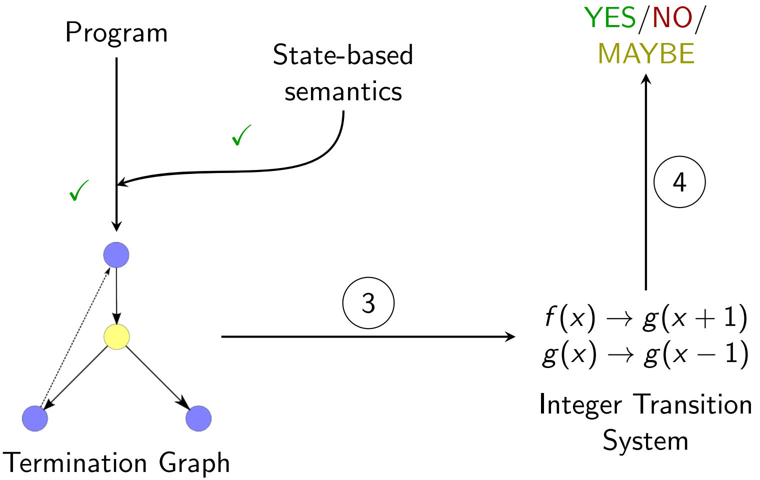
From Programs To Graphs - Final Result



Roadmap


Program
State-based semantics
✓
2
3
4
YES/NO/MAYBE
$f(x) \to g(x+1)$
$g(x) \to g(x-1)$
Integer Transition System
Termination Graph

Roadmap



YES/NO/
MAYBE
Program
State-based
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✓
✓
4
3
$f(x) \to g(x+1)$
$g(x) \to g(x-1)$
Termination Graph
Integer Transition
System

Integer Transition Systems

Integer Transition System:

$$f(x) \to f(x+1)$$

Integer Transition Systems

Integer Transition System:

$$f(x) \to f(x+1) \qquad \mid x < 0$$

From Graphs to Transition Systems

Given: Some Termination Graph

Goal: Integer Transition System that terminates if

From Graphs to Transition Systems

Given: Some Termination Graph

Goal: Integer Transition System that terminates if all runs described by the Termination Graph terminate

From Graphs to Transition Systems

Given: Some Termination Graph

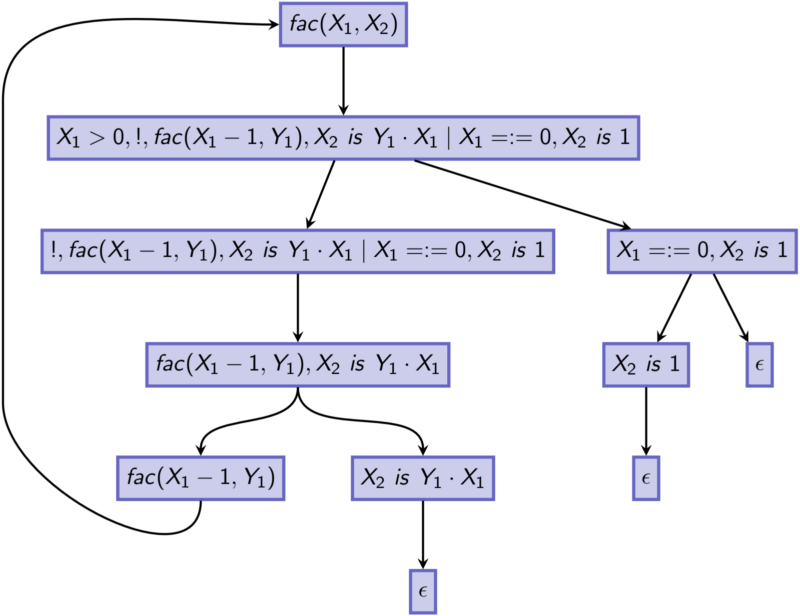
Goal: Integer Transition System that terminates if all runs described by the Termination Graph terminate

Idea: Encode graph locally, node by node

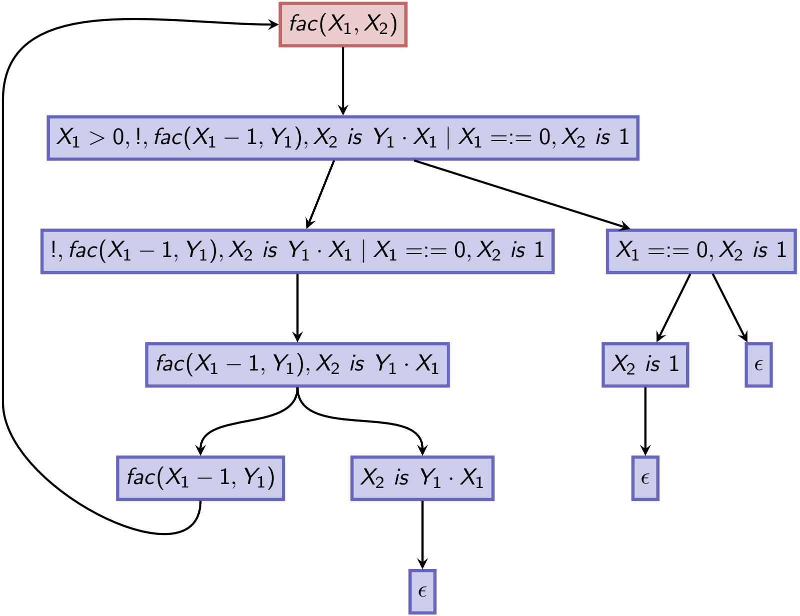
From Graphs to Transition Systems

Paths in Graph $\approx$ Evaluations

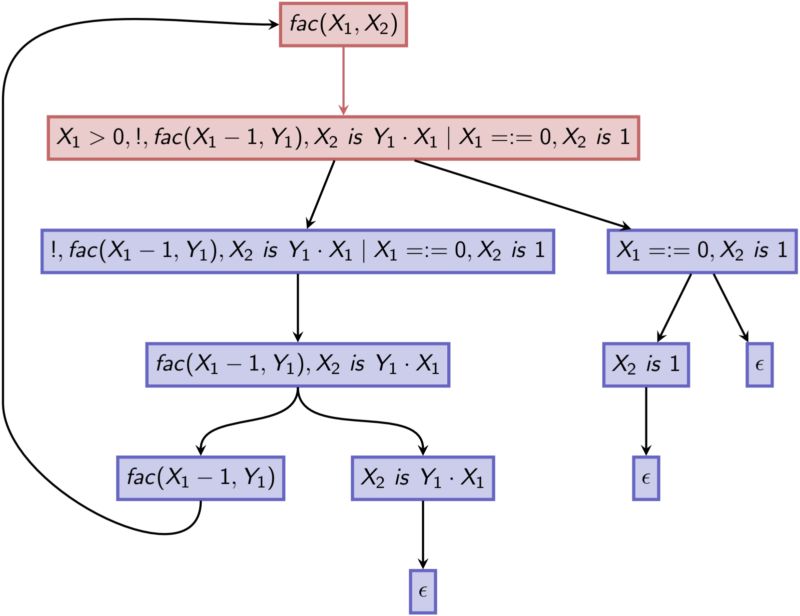
From Graphs to Transition Systems



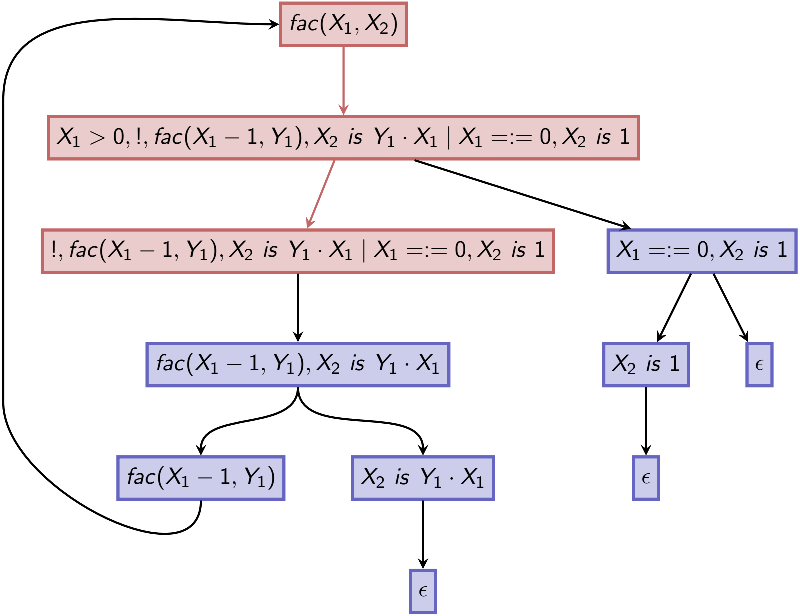
From Graphs to Transition Systems



From Graphs to Transition Systems



From Graphs to Transition Systems



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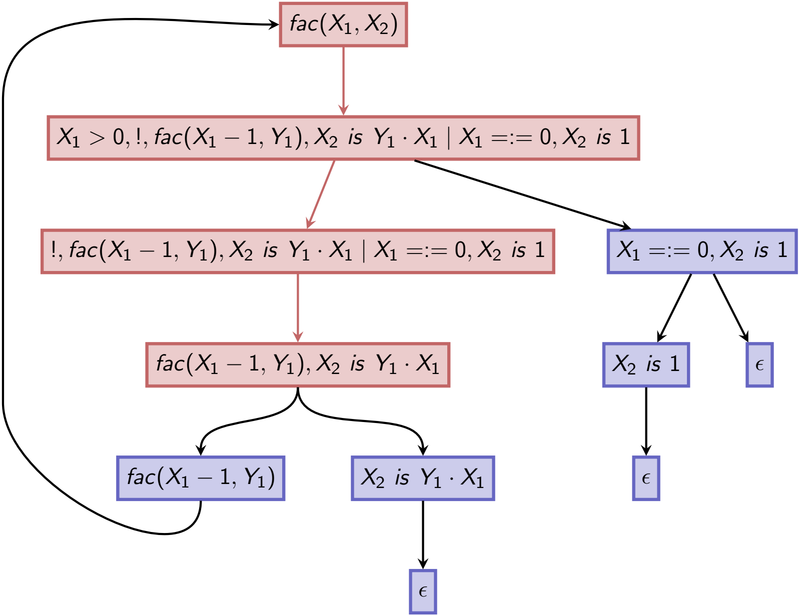
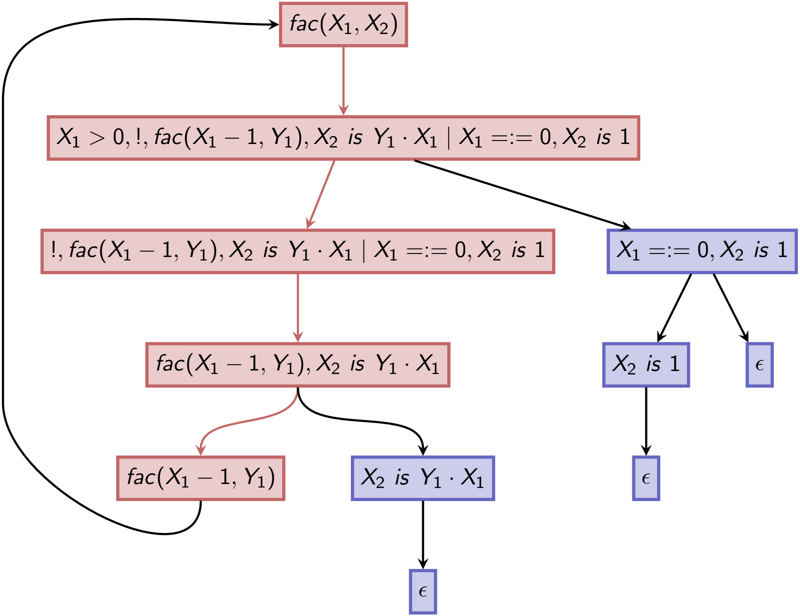
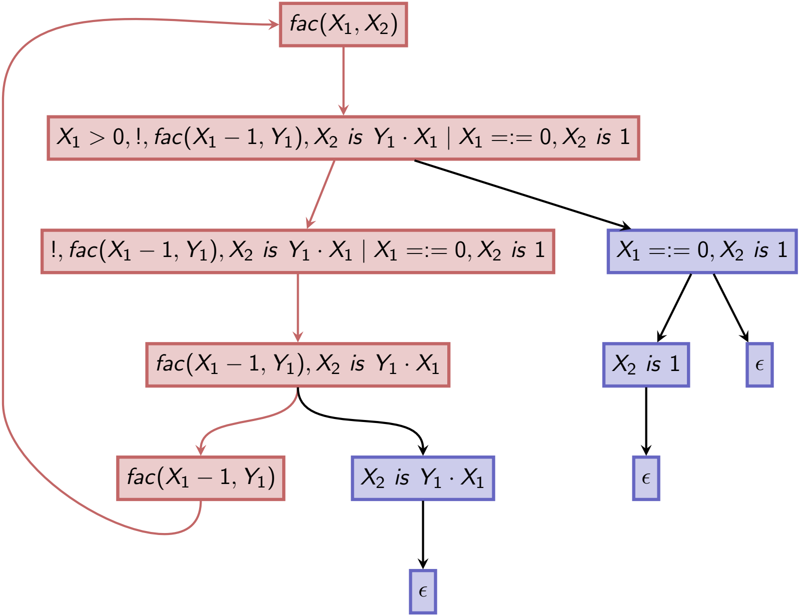


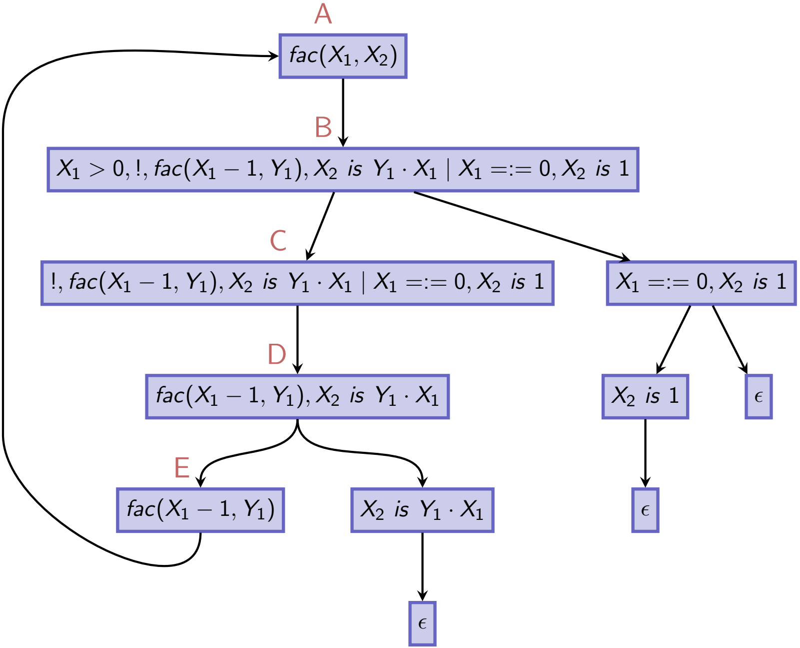
From Graphs to Transition Systems



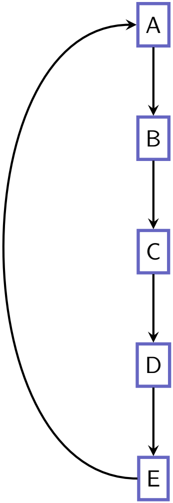
From Graphs to Transition Systems



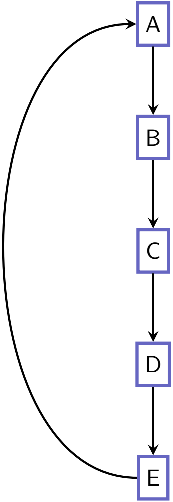
From Graphs to Transition Systems



From Graphs to Transition Systems

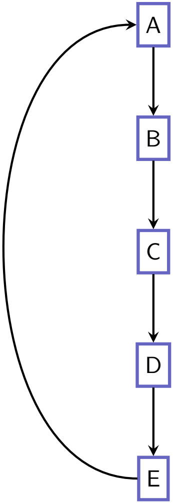


From Graphs to Transition Systems



$$\begin{array}{ccc} A & \to & B \\ B & \to & C \\ C & \to & D \\ D & \to & E \\ E & \to & A \end{array}$$

From Graphs to Transition Systems



$$\begin{array}{ccc} A & \rightarrow & B \\ B & \rightarrow & C \\ C & \rightarrow & D \\ D & \rightarrow & E \\ E & \rightarrow & A \end{array}$$

$$\downarrow$$

$$A \rightarrow A$$

From Graphs to Transition Systems

A

$$\begin{tikzcd} \mathit{fac}(\ X_1\ ,\ X_2\ ) \arrow[d, "B"] \\ X_1\ > 0, !, \mathit{fac}(\ X_1\ - 1,\ Y_1\ ),\ X_2\ \ \mathit{is}\ \ Y_1\ \cdot\ X_1\ \mid\ X_1\ =:= 0,\ X_2\ \ \mathit{is}\ 1 \end{tikzcd}$$

From Graphs to Transition Systems

$$\begin{array}{c} A \\ \boxed{fac(\boxed{X_1}, \boxed{X_2})} \\ \big\downarrow \, B \\ \boxed{\boxed{X_1} > 0, !, fac(\boxed{X_1} - 1, \boxed{Y_1}), \boxed{X_2} \ is \ \boxed{Y_1} \cdot \boxed{X_1} \mid \boxed{X_1} =:= 0, \boxed{X_2} \ is \ 1} \end{array}$$

From Graphs to Transition Systems

A

$$\begin{tikzcd}
fac(\boxed{X_1}, \boxed{X_2}) \arrow[d, "B"] \\
\boxed{X_1} > 0, !, fac(\boxed{X_1} - 1, \boxed{Y_1}), \boxed{X_2} \; is \; \boxed{Y_1} \cdot \boxed{X_1} \mid \boxed{X_1} =:= 0, \boxed{X_2} \; is \; 1
\end{tikzcd}$$

$$A(\qquad) \rightarrow B(\qquad)$$

From Graphs to Transition Systems

A

$$\boxed{fac(\boxed{X_1}, \boxed{X_2})}$$

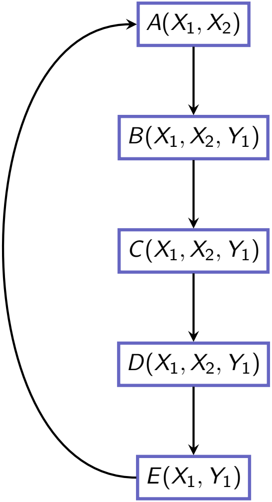
$$\downarrow$$

B

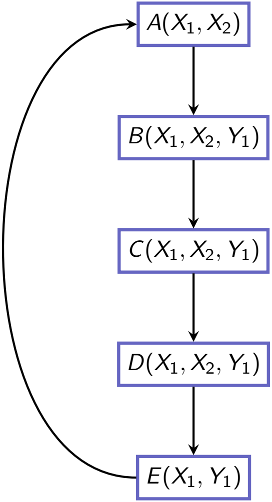
$$\boxed{\boxed{X_1} > 0, !, fac(\boxed{X_1} - 1, \boxed{Y_1}), \boxed{X_2}\ is\ \boxed{Y_1} \cdot \boxed{X_1} \mid \boxed{X_1} =:= 0, \boxed{X_2}\ is\ 1}$$

$$A(X_1, X_2) \quad \rightarrow \quad B(X_1, X_2, Y_1)$$

From Graphs to Transition Systems

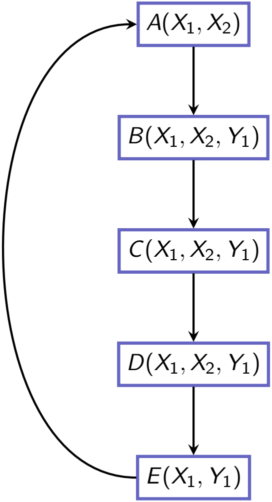


From Graphs to Transition Systems



$$
\begin{array}{lcl}
A(X_1, X_2) & \to & B(X_1, X_2, Y_1) \\
B(X_1, X_2, Y_1) & \to & C(X_1, X_2, Y_1) \\
C(X_1, X_2, Y_1) & \to & D(X_1, X_2, Y_1) \\
D(X_1, X_2, Y_1) & \to & E(X_1, Y_1) \\
E(X_1, Y_1) & \to & A(X_1, X_2)
\end{array}
$$

From Graphs to Transition Systems



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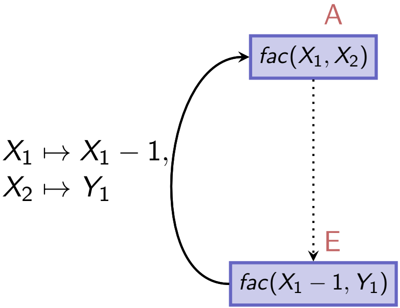
C(X_1, X_2, Y_1) & \to & D(X_1, X_2, Y_1) \\
D(X_1, X_2, Y_1) & \to & E(X_1, Y_1) \\
E(X_1, Y_1) & \to & A(X_1, X_2)
\end{array}$$

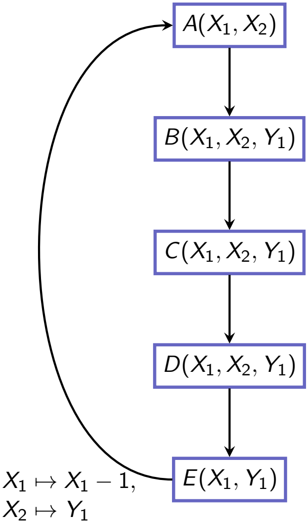
$$\downarrow$$

$$A(X_1, X_2) \quad \to \quad A(X_1, Y_1)$$

From Programs To Graphs - Instance Rule

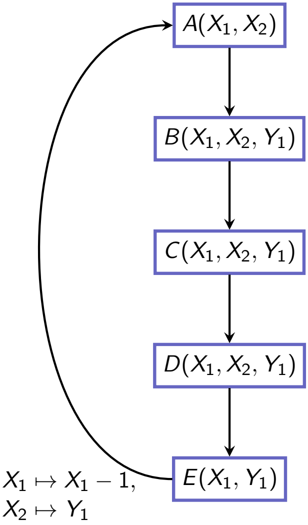


From Graphs to Transition Systems



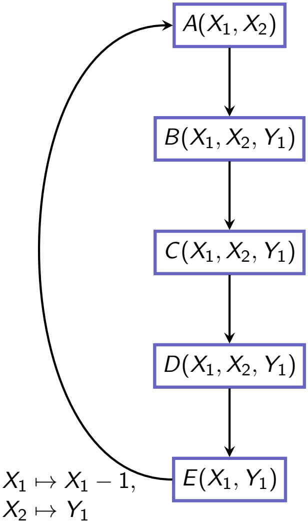
$$
\begin{array}{lcl}
A(X_1, X_2) & \to & B(X_1, X_2, Y_1) \\
B(X_1, X_2, Y_1) & \to & C(X_1, X_2, Y_1) \\
C(X_1, X_2, Y_1) & \to & D(X_1, X_2, Y_1) \\
D(X_1, X_2, Y_1) & \to & E(X_1, Y_1) \\
E(X_1, Y_1) & \to & A(\qquad\qquad)
\end{array}
$$

From Graphs to Transition Systems



$$
\begin{array}{lcl}
A(X_1, X_2) & \rightarrow & B(X_1, X_2, Y_1) \\
B(X_1, X_2, Y_1) & \rightarrow & C(X_1, X_2, Y_1) \\
C(X_1, X_2, Y_1) & \rightarrow & D(X_1, X_2, Y_1) \\
D(X_1, X_2, Y_1) & \rightarrow & E(X_1, Y_1) \\
E(X_1, Y_1) & \rightarrow & A(X_1 - 1, Y_1)
\end{array}
$$

From Graphs to Transition Systems



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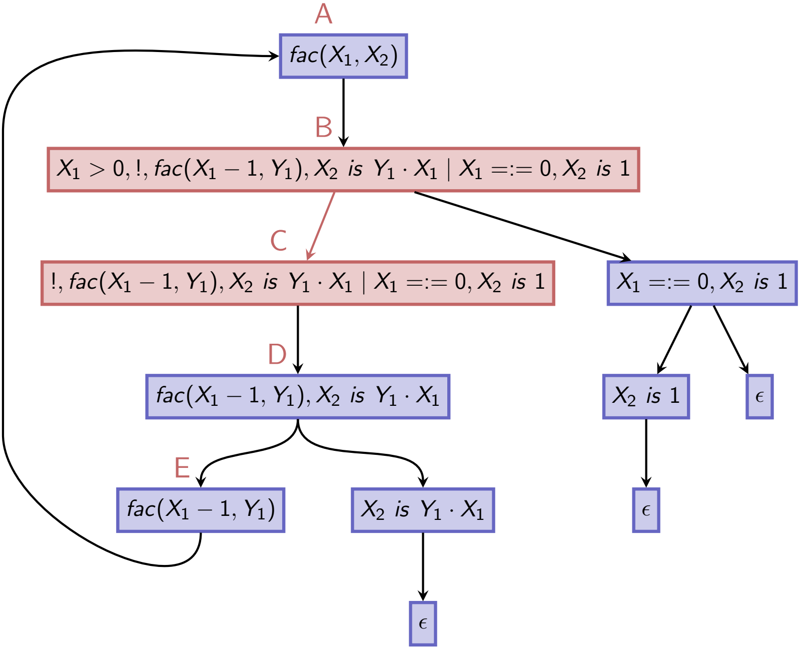
D(X_1, X_2, Y_1) & \to & E(X_1, Y_1) \\
E(X_1, Y_1) & \to & A(X_1 - 1, Y_1)
\end{array}
$$

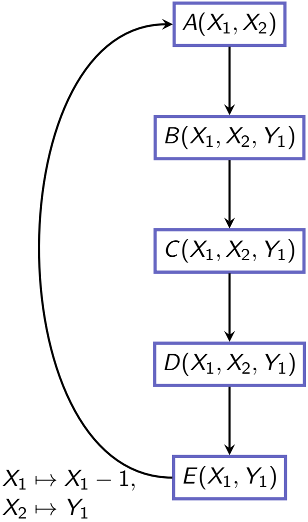
$$\downarrow$$

$$A(X_1, X_2) \to A(X_1 - 1, Y_1)$$

From Graphs to Transition Systems

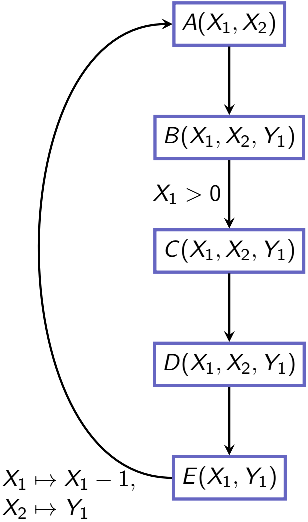


From Graphs to Transition Systems



$$
\begin{aligned}
A(X_1, X_2) &\quad\rightarrow\quad B(X_1, X_2, Y_1)\\
B(X_1, X_2, Y_1) &\quad\rightarrow\quad C(X_1, X_2, Y_1)\\
C(X_1, X_2, Y_1) &\quad\rightarrow\quad D(X_1, X_2, Y_1)\\
D(X_1, X_2, Y_1) &\quad\rightarrow\quad E(X_1, Y_1)\\
E(X_1, Y_1) &\quad\rightarrow\quad A(X_1 - 1, Y_1)
\end{aligned}
$$

From Graphs to Transition Systems



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\begin{array}{lcl}
A(X_1, X_2) & \rightarrow & B(X_1, X_2, Y_1) \\
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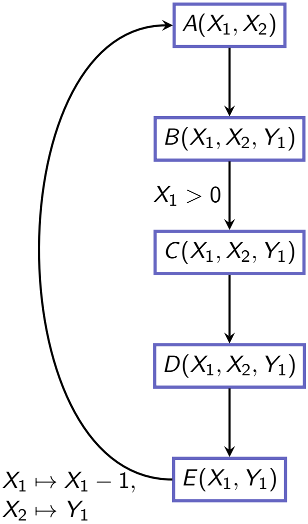
E(X_1, Y_1) & \rightarrow & A(X_1 - 1, Y_1)
\end{array}
$$

From Graphs to Transition Systems



$$
\begin{aligned}
A(X_1, X_2) &\quad\to\quad B(X_1, X_2, Y_1) \\
B(X_1, X_2, Y_1) &\quad\to\quad C(X_1, X_2, Y_1) \quad \mid X_1 > 0 \\
C(X_1, X_2, Y_1) &\quad\to\quad D(X_1, X_2, Y_1) \\
D(X_1, X_2, Y_1) &\quad\to\quad E(X_1, Y_1) \\
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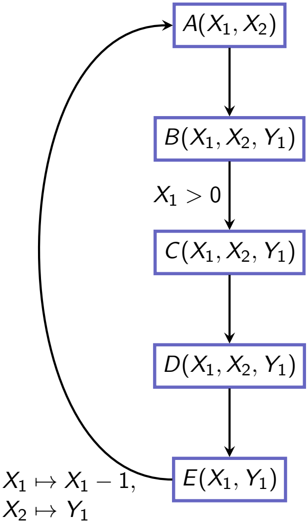
\end{aligned}
$$

From Graphs to Transition Systems



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\begin{array}{lcll}
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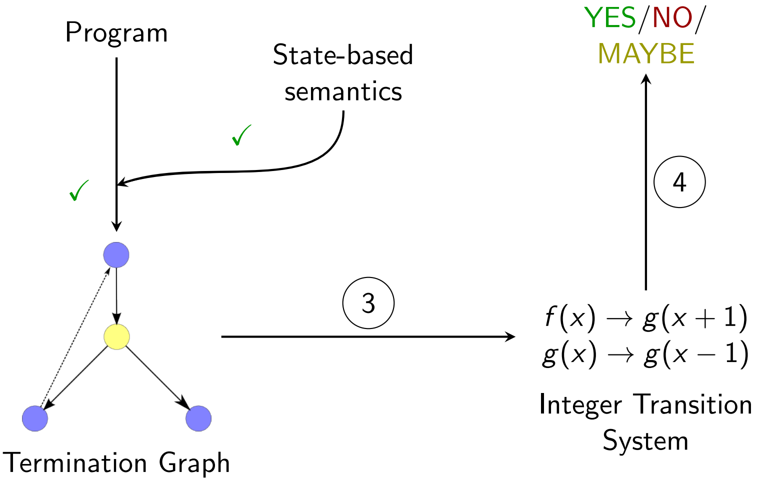
D(X_1, X_2, Y_1) & \to & E(X_1, Y_1) & \\
E(X_1, Y_1) & \to & A(X_1 - 1, Y_1) &
\end{array}
$$

$$\downarrow$$

$$A(X) \quad \to \quad A(X - 1) \quad \mid X > 0$$

Roadmap


Program
State-based semantics
YES/NO/MAYBE
4
3
$f(x) \to g(x+1)$
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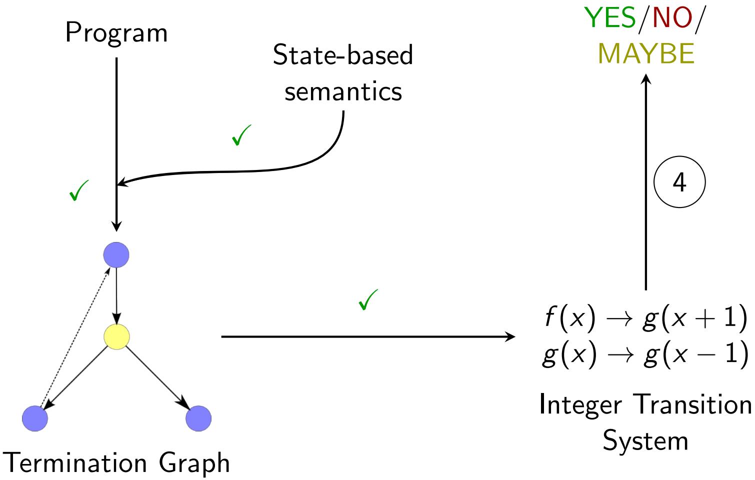
Termination Graph

Roadmap


Program
State-based
semantics
✓
✓
YES/NO/
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4
✓
$f(x) \to g(x+1)$
$g(x) \to g(x-1)$
Integer Transition
System
Termination Graph

Termination of Integer Transition Systems

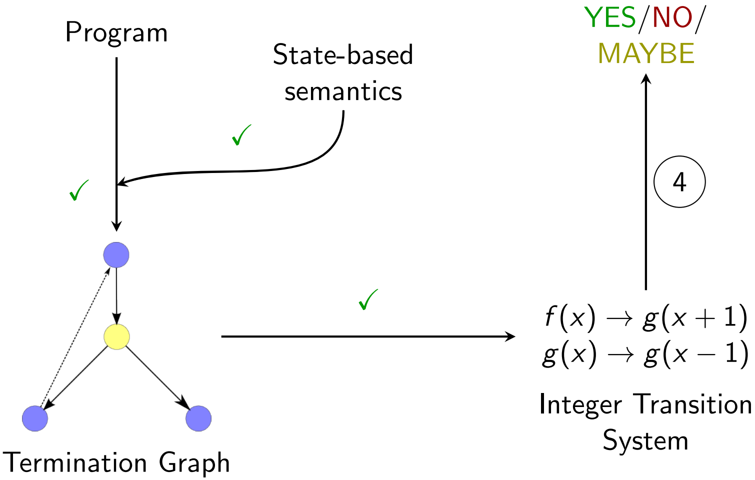
Well-studied problem

Termination of Integer Transition Systems

Well-studied problem

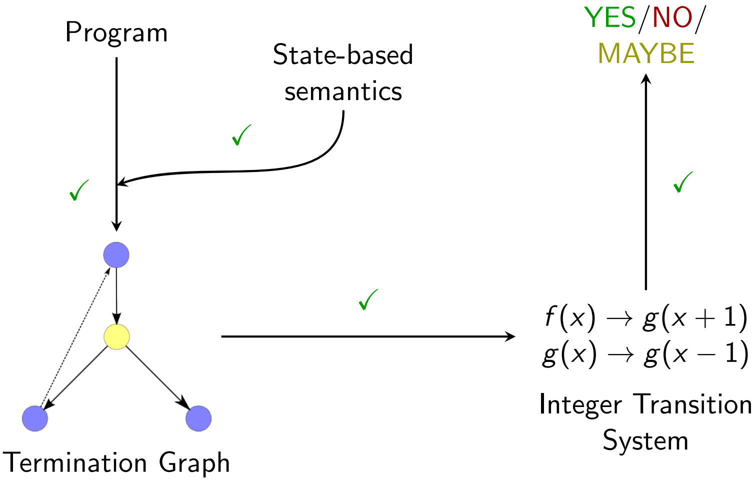
Use known techniques to show termination
(Transition Invariants, Podelski and Rybalchenko, 2004)

Roadmap

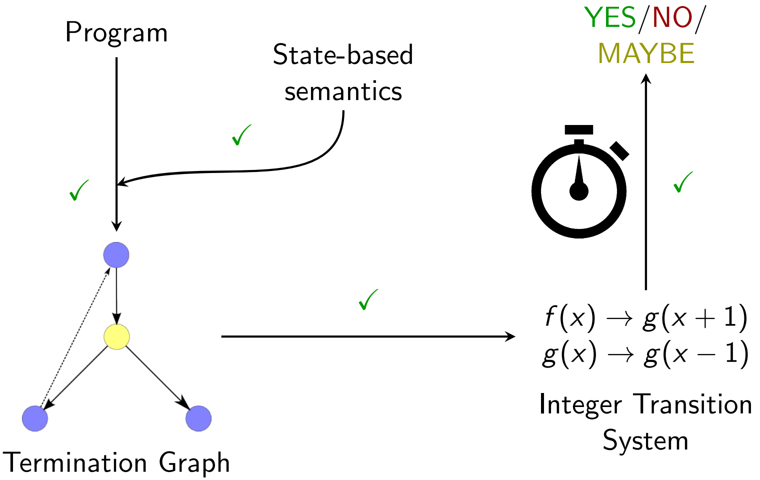


YES/NO/
MAYBE
Program
State-based
semantics
✓
✓
4
✓
$f(x) \to g(x+1)$
$g(x) \to g(x-1)$
Integer Transition
System
Termination Graph

Roadmap


YES/NO/
MAYBE
Program
State-based
semantics
✓
✓
✓
✓
$f(x) \to g(x+1)$
$g(x) \to g(x-1)$
Integer Transition
System
Termination Graph

Roadmap



Stopwatch: `http://icons8.com/`

Results - Compared Approaches

Term Rewriting with Argument Filter Directly from Program

(Termination Proofs, Schneider-Kamp et al., 2009)

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(Analysis of Arithmetic Prolog Programs, Weinert, 2015)

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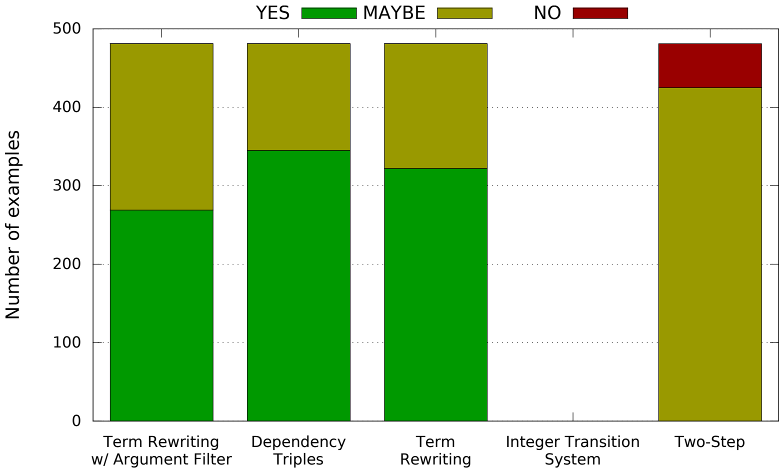
Two-Step Constraint Satisfaction Problem
(Non-termination Analysis, Voets et al., 2011)

Results - Logic Benchmarks - Power



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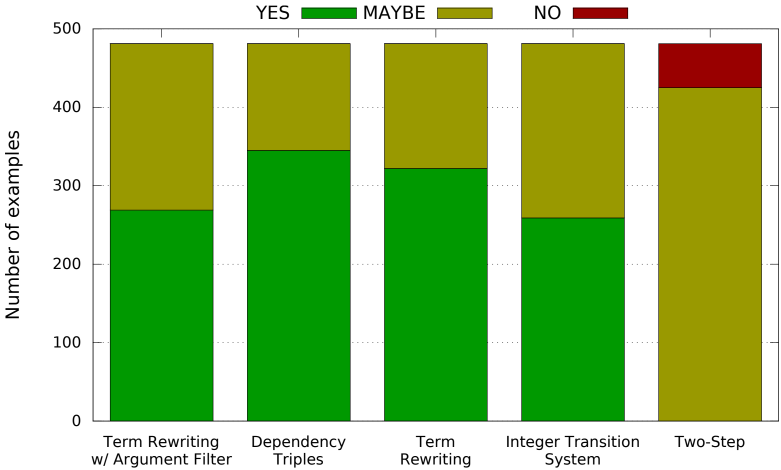
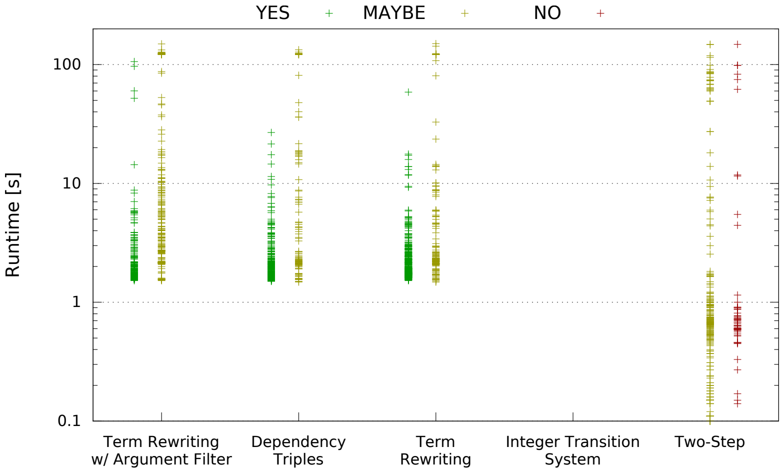


Results - Logic Benchmarks - Runtime



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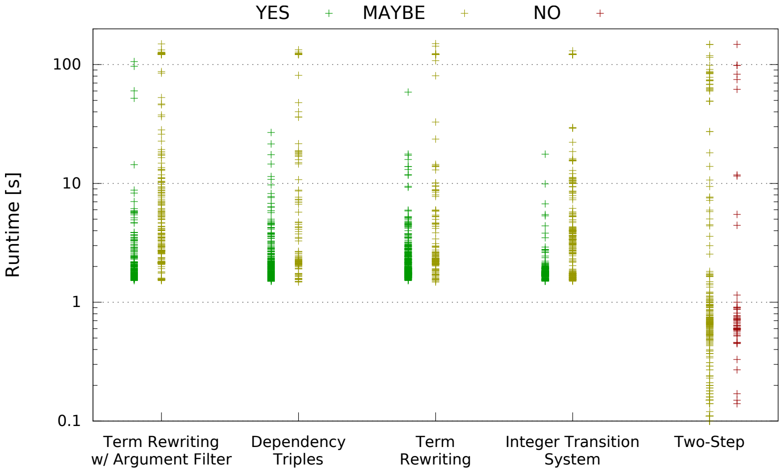
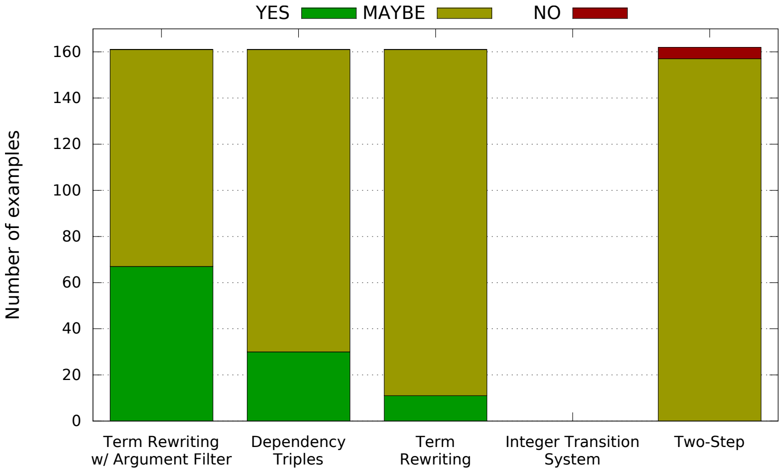
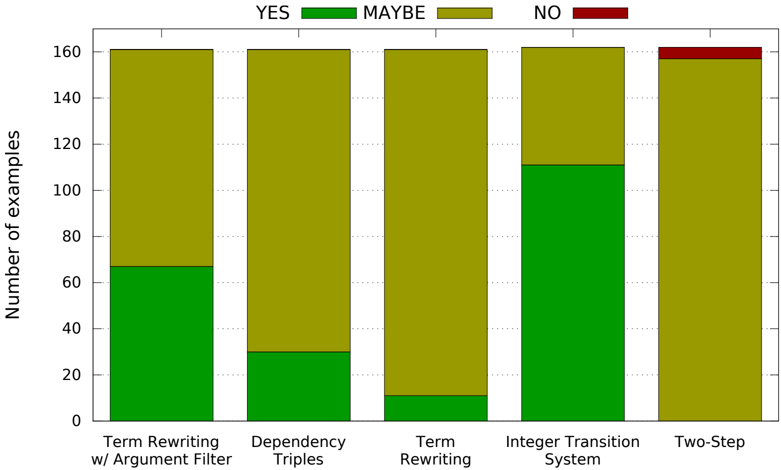


Results - Numerical Benchmarks - Power



Results - Numerical Benchmarks - Power



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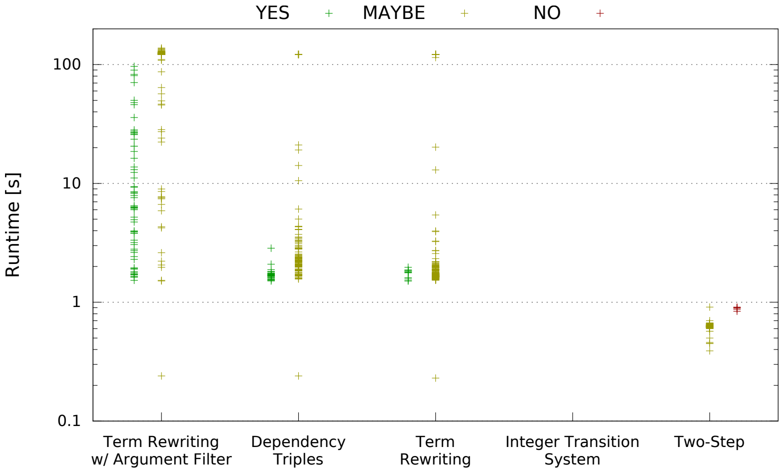

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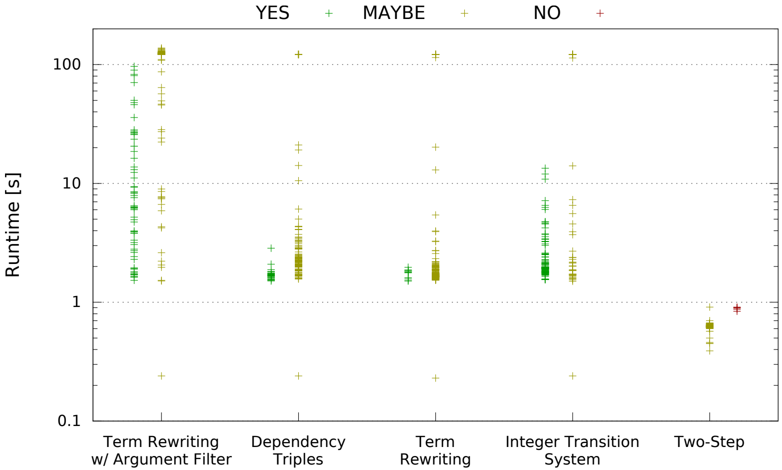

Contributions - Theoretical

- Extended construction of Termination Graphs, taking arithmetic comparisons and evaluations into account
- Developed new construction of Integer Transition System from Termination Graphs

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- Proved soundness of all steps of the construction

Contributions - Practical

- Implemented extension of construction of Termination Graphs, optimization through SMT solver
- Implemented construction of Integer Transition Systems from Termination Graphs

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- Implemented extension of construction of Termination Graphs, optimization through SMT solver
- Implemented construction of Integer Transition Systems from Termination Graphs
- Added 162 numerical benchmarks to benchmark suite
- Performed experiments comparing this approach to existing ones

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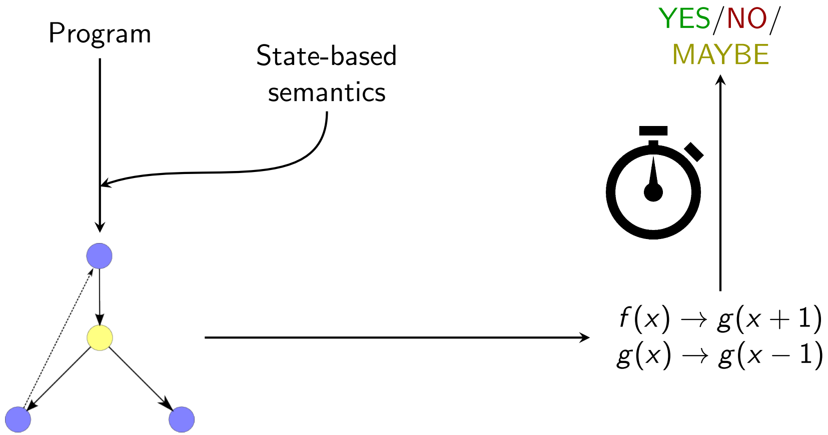
`alexander.weinert@rwth-aachen.de`



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