# Parity Games with Weights

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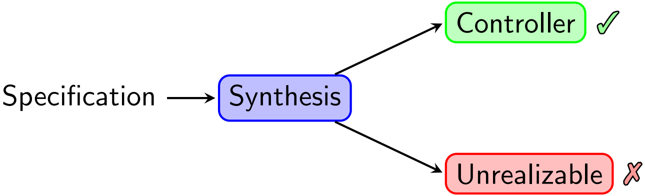
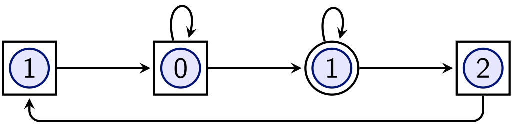
# Controller Synthesis




In most cases boils down to solving parity game

# Parity Games



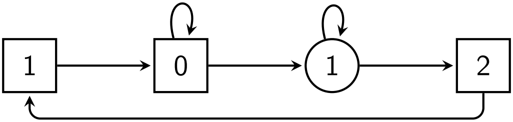

$1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 1 \longrightarrow 1 \longrightarrow 2 \rightarrow \cdots$

Max. color inftly often even

## Proposition (Jurdziński 1998, Calude et al. 2017)

*Solving parity games is in* $\mathrm{UP} \cap \mathrm{CO\text{-}UP}$ *and they can be solved in quasi-polynomial time. Both players have memoryless winning strategies.*

# Finitary Parity Games



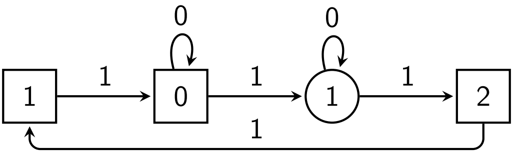

$$\underbrace{1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2}_{\text{3 steps}} \longrightarrow \underbrace{1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 1 \longrightarrow 1 \longrightarrow 2}_{\text{5 steps}} \longrightarrow \cdots$$

Aim for Player 0:
Eventually bound steps between requests and answers

## Proposition (Chatterjee, Henzinger, and Horn, 2009)

*Solving finitary parity games is in* PTime. *Player* 0 *has memoryless winning strategies.*

# Parity Games with Costs




$$1 \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow 0 \rightarrow 0 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$$

$$\underbrace{1 \rightarrow 0 \rightarrow 1 \rightarrow 2}_{\text{Weight 3}} \rightarrow \underbrace{1 \rightarrow 0 \rightarrow 0 \rightarrow 1 \rightarrow 1 \rightarrow 2}_{\text{Weight 3}} \rightarrow \cdots$$

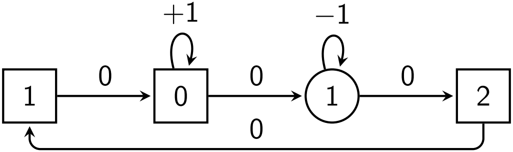
Aim for Player 0:

Eventually bound ~~steps~~ weight between request and answer

## Proposition (Fijalkow and Zimmermann 2014 / Mogavero, Murano, and Sorrentino 2015)

*Solving parity games with costs is in* $\mathrm{UP} \cap \mathrm{co}\text{-}\mathrm{UP}$. *Player* 0 *has memoryless winning strategies.*

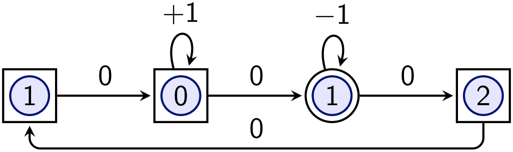
# Parity Games with Weights

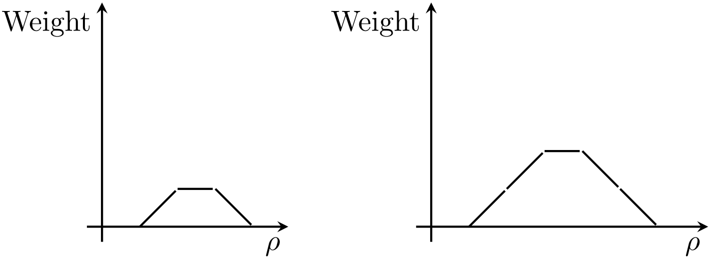



$$\underbrace{1 \longrightarrow 0 \longrightarrow 1 \longrightarrow 2}_{\text{Weight } 0} \longrightarrow \underbrace{1 \longrightarrow 0 \longrightarrow 0 \longrightarrow 1 \longrightarrow 1 \longrightarrow 2}_{\text{Weight } 0} \rightarrow \cdots$$

Not the complete picture. . .

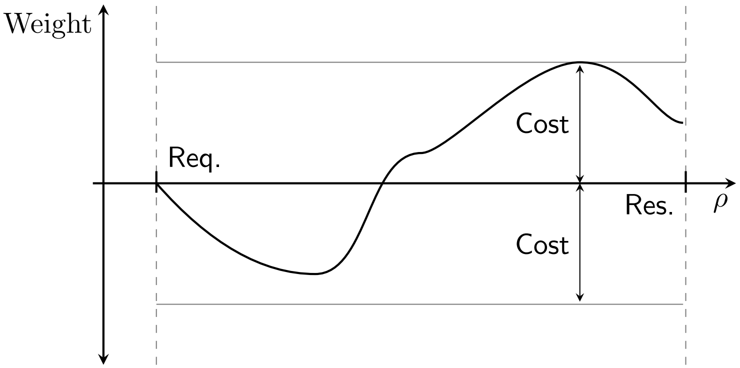
# Cost Measure




Intuitively:
Measure size of resource required to answer request

# Cost Measure




Cost of answering: **Amplitude on path to answer**

# Outline

- Definition ✓
- Complexity
- Memory
- Bounds

# Solving Parity Games with Weights

**Problem:** Given a parity game with weights $\mathcal{G}$ and a vertex $v$ of $\mathcal{G}$, does Player 0 win $\mathcal{G}$ from $v$?

**Problem:** Given an arena $\mathcal{A}$ with colors and weights, can Player 0 bound the costs of answering odd colors with higher even ones?

**Proof idea:** Analogously to (Fijalkow and Zimmermann, 2014)

1. Strengthen parity condition with weights
2. Reduce to solving energy parity games

# Bounding Parity Games with Weights

**First Step:** Strengthen parity condition with weights → bounded parity games with weights.

**Parity with weights:** Player 1 can win by causing infinitely many requests with infinite cost

**Bounded parity with weights:** Player 1 can win by causing a single request with infinite cost

## Lemma

*Parity games with weights can be solved by solving polynomially many bounded parity games with weights of polynomial size.*

# Solving Bounded Parity Games with Weights

## Lemma

*Parity games with weights can be solved by solving polynomially many bounded parity games with weights of polynomial size.*

**Next step:** Solving bounded parity games with weights via energy parity games.

## Lemma

*Bounded parity games with weights can be solved by solving polynomially many energy parity games of polynomial size.*

## Proposition (Chatterjee and Doyen, 2012)

*Solving energy parity games is in* $\mathrm{NP} \cap \mathrm{co\text{-}NP}$.

## Theorem

*Solving parity games with weights is in* $\mathrm{NP} \cap \mathrm{co\text{-}NP}$.

# Solving Bounded Parity Games with Weights

## Proposition (Daviaud et al., 2018)

*Energy parity games can be solved in quasi-pseudo-polynomial time.*

## Corollary (Daviaud et al., 2018)

*Parity games with weights can be solved in quasi-pseudo-polynomial time.*

# There and Back Again

**So far:** Solved parity games with weights via iteratively solving energy parity games.

**Also possible:** Solving energy parity games via iteratively solving parity games with weights.

**Proof idea:** Again detour via bounded parity games with weights.

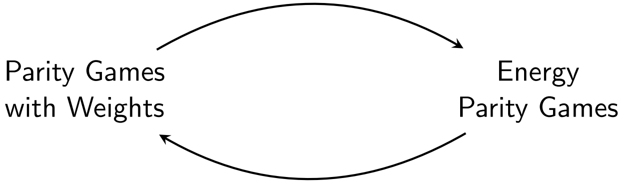
## Lemma

*Energy parity games can be solved by solving polynomially many parity games with weights of polynomial size.*

# There and Back Again

## Theorem

*The problems of solving parity games with weights and of solving energy parity games are polynomial-time equivalent.*



# Outline

- Definition ✓
- Complexity ✓
- Memory
- Bounds

# Memory Requirements (Upper Bound)

**Known so far:** Bounded parity games with weights can be solved by solving polynomially many energy parity games.

**Intuition:**

1. Each energy parity game $\mathcal{G}'_v$ is "tasked" with answering the request posed at vertex $v$ with bounded cost.
2. Player 0 wins $\mathcal{G}$ by simulating winning $\mathcal{G}'_v$ for vertex carrying largest requested color.

# Memory Requirements (Upper Bound)

## Proposition (Chatterjee and Doyen, 2012)

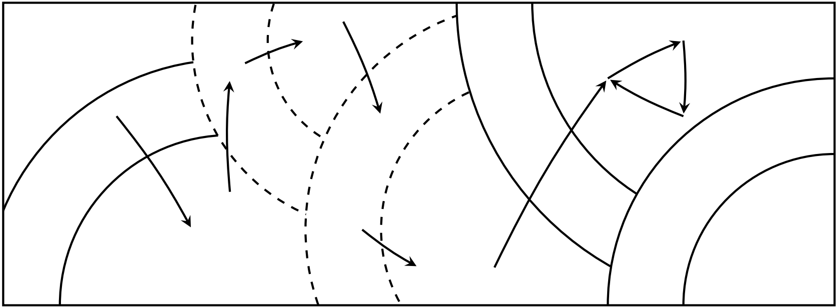
*In an energy parity game with $n$ vertices, $d$ odd colors, and largest absolute weight $W$, memory of size $\mathcal{O}(ndW)$ suffices for her to implement a winning strategy from her winning region.*

## Lemma

*In a bounded parity game with weights with $n$ vertices, $d$ odd colors, and largest absolute weight $W$, memory of size $\mathcal{O}(nd^2W)$ suffices for Player $0$ to implement a winning strategy for her from her winning region.*

# Memory Requirements (Upper Bound)

**Known so far:** Parity games with weights can be solved by solving polynomially many bounded parity games with weights.



**Winning Strategy:** At every "border", restart winning bounded parity game with weights

# Memory Requirements (Upper Bound)

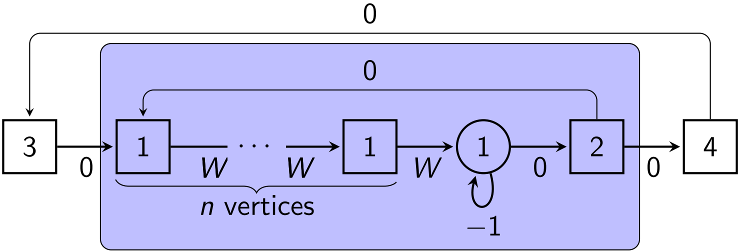
**Lemma**

*In a bounded parity game with weights with $n$ vertices, $d$ odd colors, and largest absolute weight $W$, memory of size $\mathcal{O}(nd^2W)$ suffices for Player $0$ to implement a winning strategy for her from her winning region.*

**Theorem**

*In a parity game with weights with $n$ vertices, $d$ odd colors, and largest absolute weight $W$, memory of size $\mathcal{O}(nd^2W)$ suffices for her to implement a winning strategy from her winning region.*

# Memory Requirements (Lower Bound)




## Theorem

*Player 0 requires $nW + 1$ memory states to implement a winning strategy in the above game.*

# Outline

- Definition ✓
- Complexity ✓
- Memory ✓
- Bounds

# Weight Bounds (Upper Bounds)

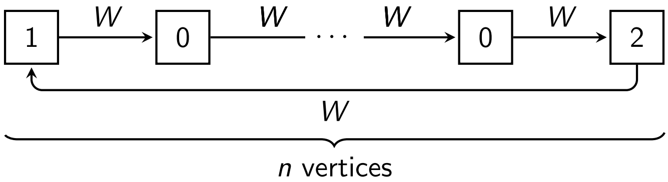
## Proposition (Chatterjee and Doyen, 2012)

*In an energy parity game $\mathcal{G}$ with $n$ vertices, $d$ odd colors, and largest absolute weight $W$, if Player $0$ wins $\mathcal{G}$ from some vertex $v$, then she has a strategy that bounds the energy level from below by $-nW$.*

## Theorem

*In a parity game with weights $\mathcal{G}$ with $n$ vertices, $d$ odd colors, and largest absolute weight $W$, if Player $0$ wins $\mathcal{G}$ from some vertex $v$, the she has a strategy that bounds the cost of plays by some value in $\mathcal{O}((ndW)^2)$.*

# Weight Bounds (Lower Bounds)




**Theorem**

*Player 0 wins the above game, but every play has a cost of $(n-1)W$.*

# Conclusion

| | Finitary | Costs | Weights |
|---|---|---|---|
| Complexity | PTIME | UP ∩ co-UP | |
| Memory | positional | positional | $\mathcal{O}(nd^2W)$ |
| Bounds | $n$ | $nW$ | $\mathcal{O}((ndW)^2)$ |

**Future Work:**

- Tighten bounds
  - Lower bound on memory: $nW + 1$
  - Lower bound on cost: $nW$
- Different cost measures
- Multidimensional parity games with weights
- Threshold problem: Enforce cost at most $b$ (EXPTIME-complete)