Quantitative Reductions and Vertex-Ranked Games

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Overview

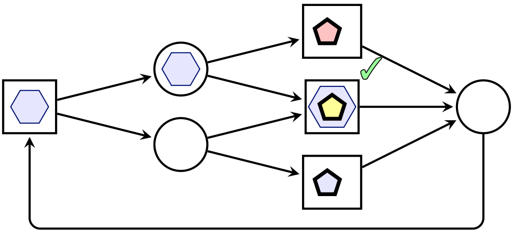
In General: Game reductions are popular for solving infinite games

Recently: Trend towards quantitative games

Problem: Reductions ignore quantitative features

Solution: Lift reduction to quantitative setting

Reachability Games



Winning condition: Play reaches either  or  or 

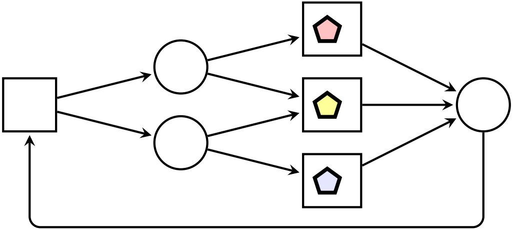
Reachability games can be effectively solved

The Big Picture

Reachability

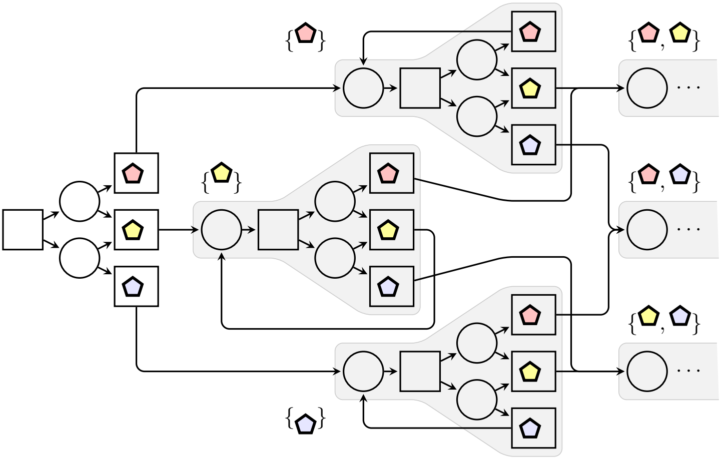
✓

Generalized Reachability: The Problem



Winning condition:
Reach one from {, } and one from {, }.

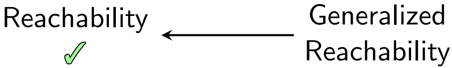
Generalized Reachability: One Solution



Winning condition: Reach some memory state S with $S \cap \{⬠, ⬠\} \neq \emptyset$ and with $S \cap \{⬠, ⬠\} \neq \emptyset$

Reachability Condition

The Big Picture



Reductions

Let $\mathcal{G}$ and $\mathcal{G}'$ be games.

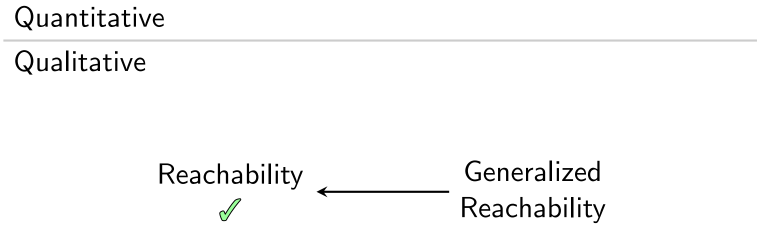
$$\mathcal{G} \leq_M \mathcal{G}'$$

iff

- $\mathcal{G}'$ is $\mathcal{G}$ together with memory structure.
- Winning condition is carried over from $\mathcal{G}$ to $\mathcal{G}'$ (modulo $\mathcal{M}$).

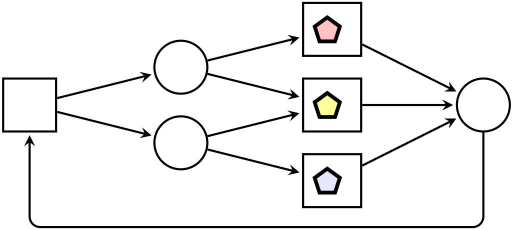
Theorem
If Player 0 wins $\mathcal{G}'$, then she wins $\mathcal{G}$ using $\mathcal{M}$.

The Big Picture



Quantitative Generalized Reachability

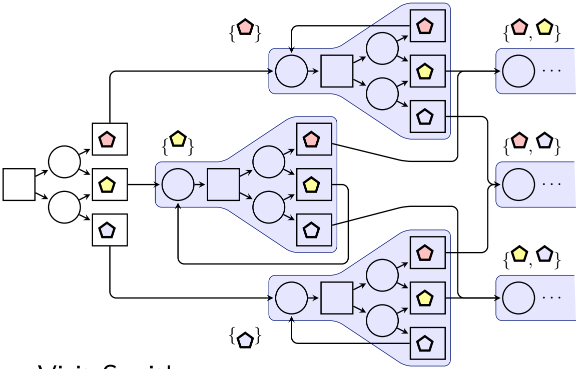
Assign cost to each play.



$$\mathrm{Cst}(\rho) = \begin{cases} 0 & \text{if } \{ \cdot \} \text{ and } \{ \cdot \} \text{ are visited} \\ 1 & \text{if one of them is visited} \\ 2 & \text{if neither is visited} \end{cases}$$

Set 1:  Set 2: 

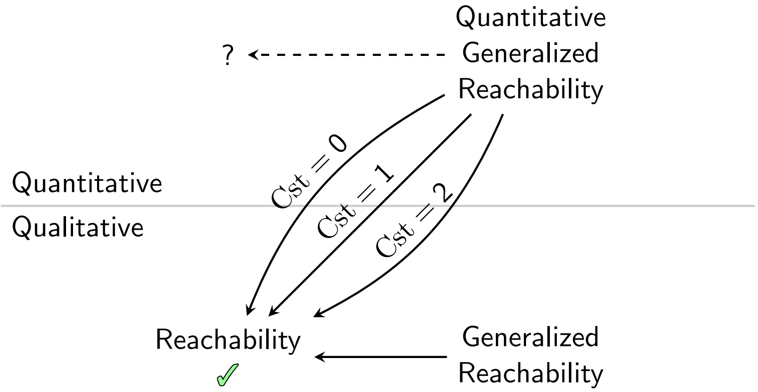
Quant. Gen. Reach.: Three Solutions



Win = Visit S with

$$\begin{cases} \{⬠_{red}, ⬠_{yellow}\} \subseteq S \text{ and } \{⬠_{yellow}, ⬠_{blue}\} \subseteq S & \text{for } \mathrm{Cst} = 0 \\ \{⬠_{red}, ⬠_{yellow}\} \subseteq S \text{ or } \{⬠_{yellow}, ⬠_{blue}\} \subseteq S & \text{for } \mathrm{Cst} = 1 \\ S \text{ arbitrary} & \text{for } \mathrm{Cst} = 2 \end{cases}$$

The Big Picture



Quantitative Reductions

Let $\mathcal{G}$ and $\mathcal{G}'$ be quantitative games.

$$\mathcal{G} \leq_M \mathcal{G}'$$

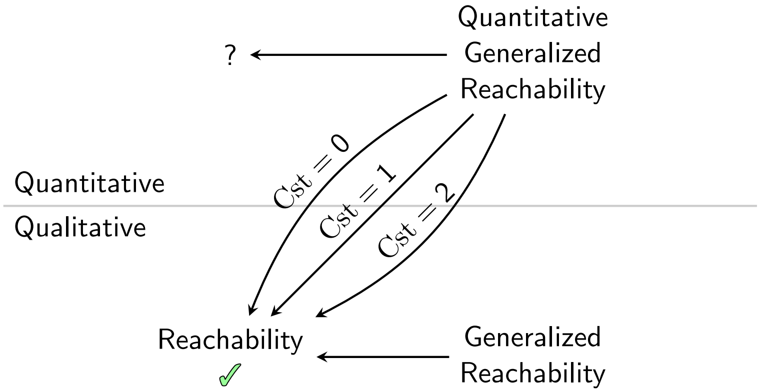
iff

- $\mathcal{G}'$ is ($\mathcal{G}+$ memory structure) and
- ~~Winning condition~~ Cost function is carried over from $\mathcal{G}$ to $\mathcal{G}'$ (modulo $\mathcal{M}$).

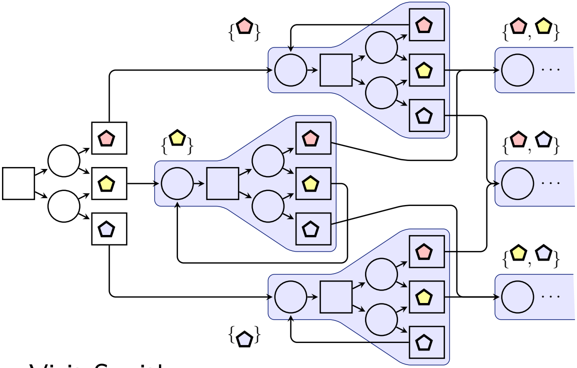
Theorem

If Player 0 can ~~win~~ enforce $\mathrm{Cst} \leq b$ in $\mathcal{G}'$, then she can ~~win~~ enforce $\mathrm{Cst} \leq b$ in $\mathcal{G}$ using $\mathcal{M}$.

The Big Picture



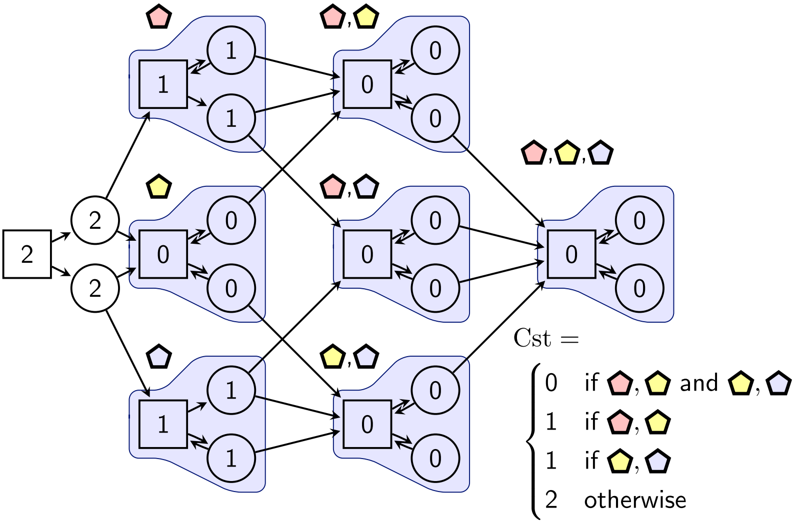
Recap: Quant. Gen. Reach.



$Win =$ Visit S with

$$\begin{cases} \{⬠_{\text{red}}, ⬠_{\text{yellow}}\} \subseteq S \text{ and } \{⬠_{\text{yellow}}, ⬠_{\text{blue}}\} \subseteq S & \text{for } \mathrm{Cst} = 0 \\ \{⬠_{\text{red}}, ⬠_{\text{yellow}}\} \subseteq S \text{ or } \{⬠_{\text{yellow}}, ⬠_{\text{blue}}\} \subseteq S & \text{for } \mathrm{Cst} = 1 \\ S \text{ arbitrary} & \text{for } \mathrm{Cst} = 2 \end{cases}$$

Quantitative Reduction



Vertex-Ranked Games

Ingredients:

- Vertex set V,
- Winning Condition $\mathrm{Win} \subseteq V^\omega$,
- ranking $\Omega \colon V \to \mathbb{N}$.

Vertex-Ranked lim sup-condition:

$$\textsc{Rank}^{\mathsf{lim}}(\mathrm{Win}, \textsc{Rank})\colon$$

$$v_0 v_1 v_2 \cdots \mapsto \begin{cases} \limsup_{j \to \infty} \textsc{Rank}(v_j) & \text{if } v_0 v_1 v_2 \cdots \in \mathrm{Win} \\ \infty & \text{otherwise} \end{cases},$$

Solving Vertex-Ranked lim sup-Games

Given: Vertex-Ranked lim sup-game $\mathcal{G}$, vertex v, threshold b

Question: Can Player 0 ensure cost at most b from v?

Solution Idea: Reduce solving to iteratively solving easier game.

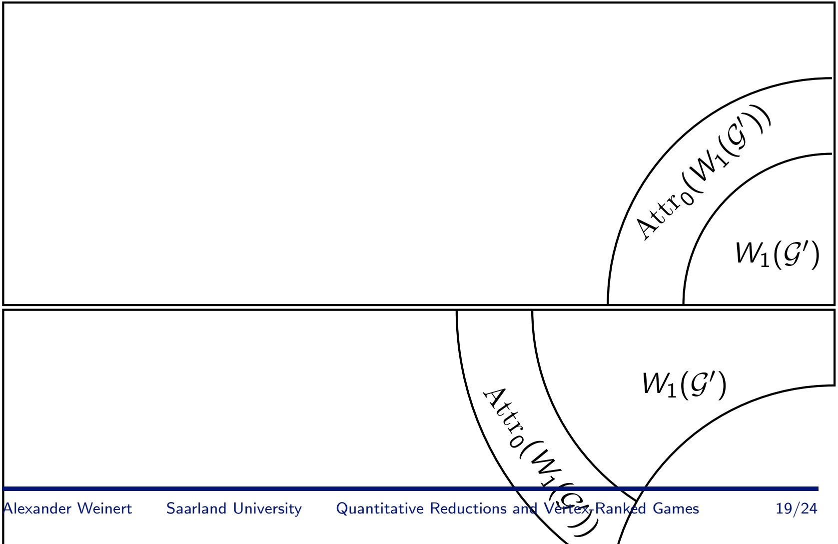
Vertex-Ranked sup-condition:

$$\textsc{Rank}^{\mathsf{lim}}(\mathrm{Win}, \textsc{Rank}):$$

$$v_0 v_1 v_2 \cdots \mapsto \begin{cases} \sup_{j \to \infty} \textsc{Rank}(v_j) & \text{if } v_0 v_1 v_2 \cdots \in \mathrm{Win} \\ \infty & \text{otherwise} \end{cases},$$

Solving Vertex-Ranked lim sup-Games

Given: Vertex-ranked lim sup-game $\mathcal{G}$, corresponding sup-game $\mathcal{G}'$.



Solving Vertex-Ranked sup-Games

Given: Vertex-Ranked $\limsup$-game $\mathcal{G}$, vertex v, threshold b

Question: Can Player 0 ensure cost at most b from v?

Solution Idea:

1. Find vertices of rank $> b$,
2. remove 1-attractor of those vertices,
3. solve resulting game qualitatively.

Theorem

Vertex-Ranked sup-*games can be solved with only an additive overhead over the qualitative solution.*

Putting it all Together

Theorem

Vertex-ranked $\limsup$-*game can be solved by solving polynomially many* sup-*games with the same winning condition*

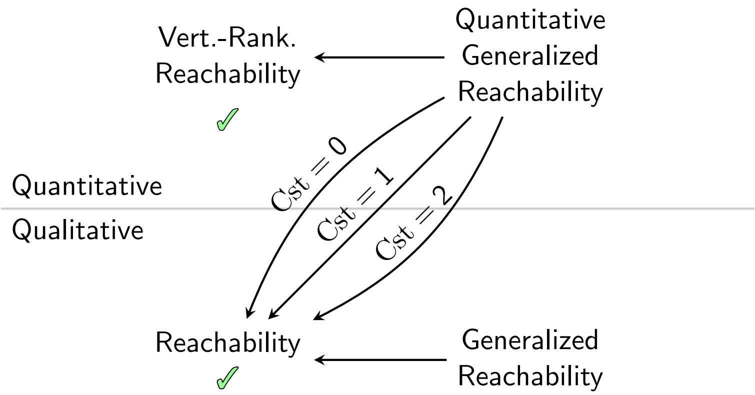
Theorem

Vertex-ranked sup-*games can be solved with only an additive overhead over the qualitative solution.*

Corollary

Vertex-ranked $\limsup$-*game can be solved with only a polynomial overhead*

The Big Picture



In the Paper

- Strategies witnessing results for vertex-ranked games
- Quantitative Reductions for Request-Response games
- Fault resilient strategies

Conclusion

Contribution

- Lifted reductions to quantitative games
- Solved wide range of general-purpose quantitative games

Next Steps