Quantitative Reductions and Vertex-Ranked Games

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Overview

In General: Game reductions are popular for solving infinite games

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Recently: Trend towards quantitative games

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Problem: Reductions ignore quantitative features

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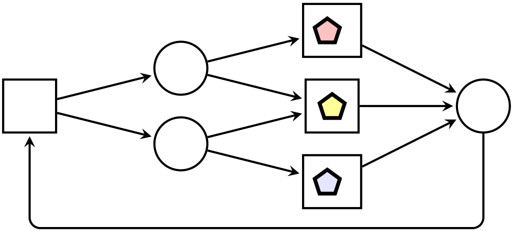
Recently: Trend towards quantitative games

Problem: Reductions ignore quantitative features

Solution: Lift reduction to quantitative setting

Reachability Games



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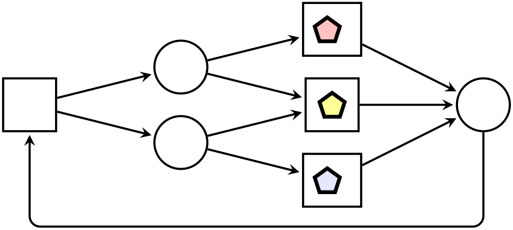
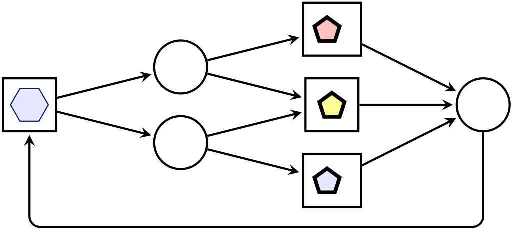


Winning condition: Play reaches either  or  or 

Reachability Games



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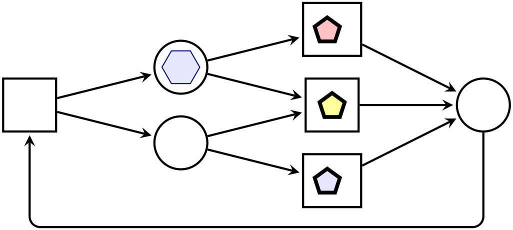

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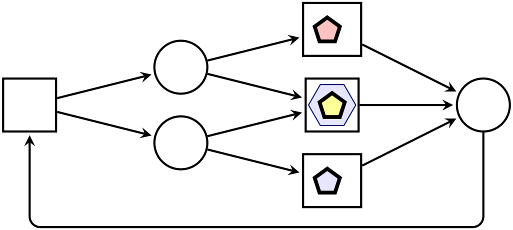

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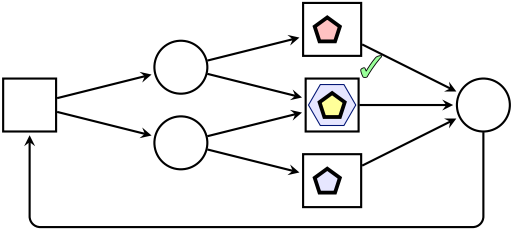

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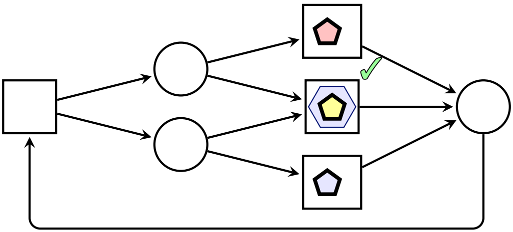

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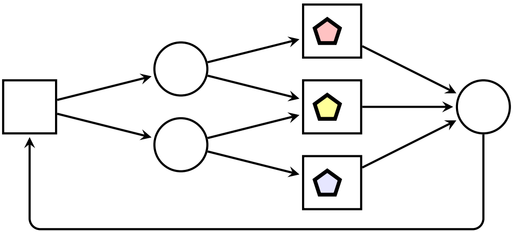
Reachability games can be effectively solved

The Big Picture

Reachability

✓

Generalized Reachability: The Problem



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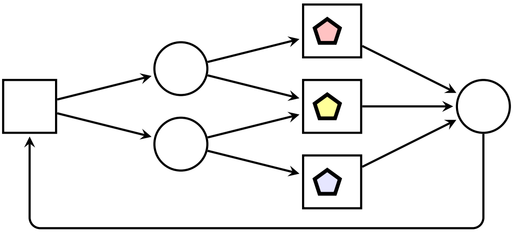
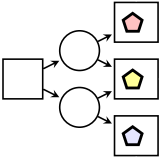


Winning condition:
Reach one from {, } and one from {, }.

Generalized Reachability: One Solution



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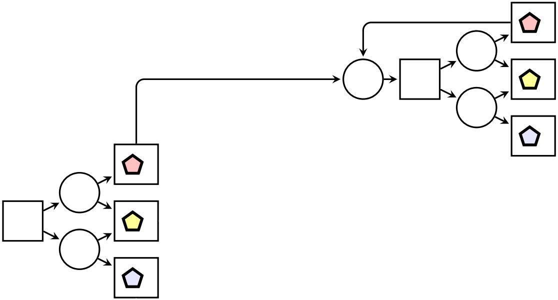

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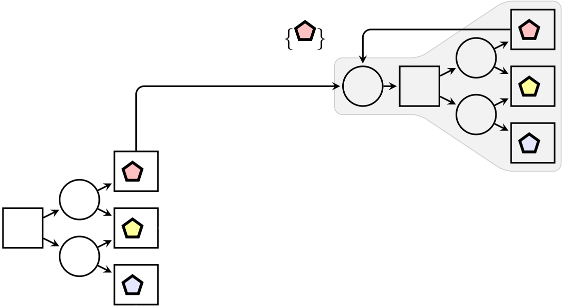

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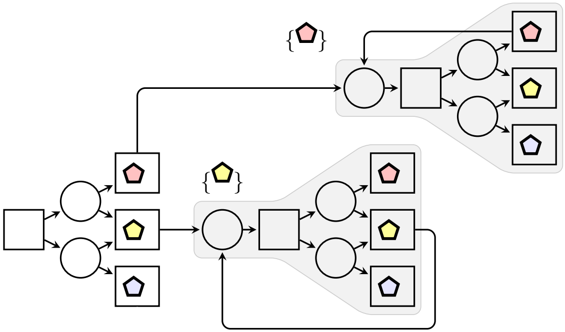

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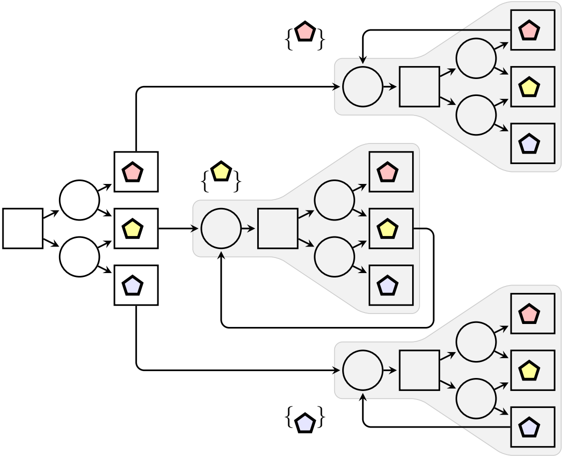

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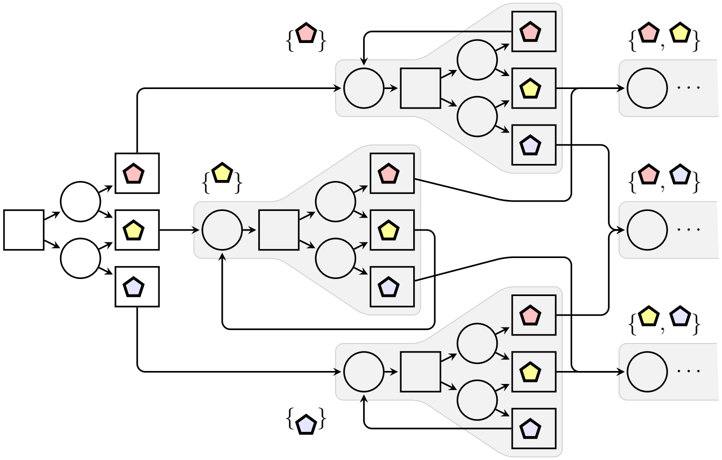

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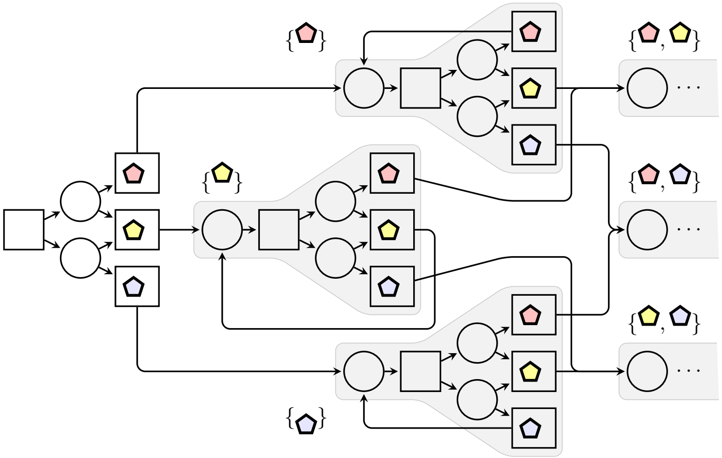

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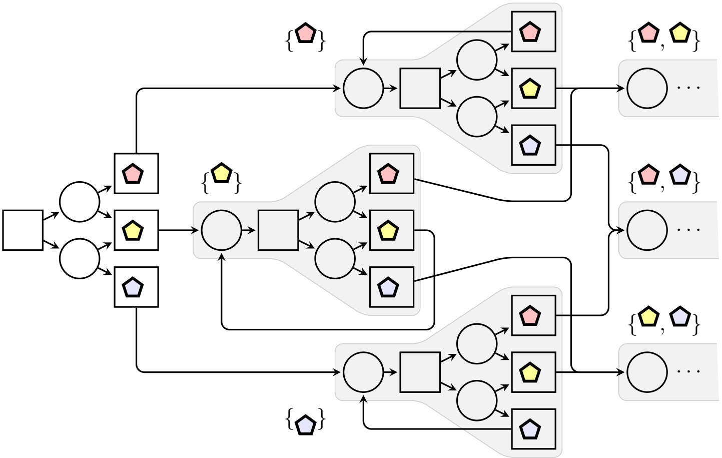

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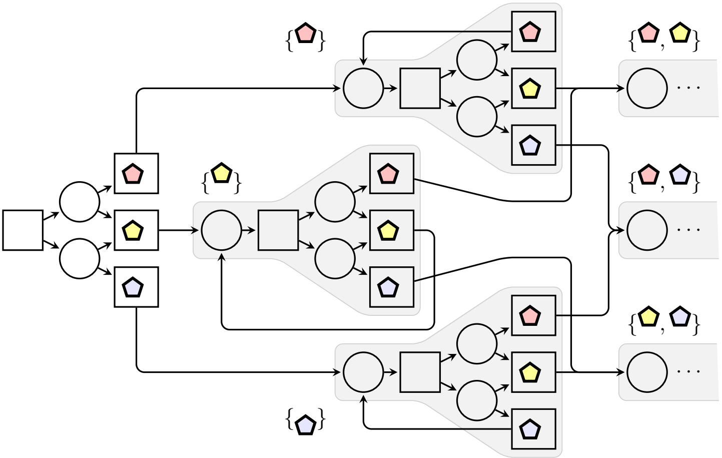

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Reachability Condition

The Big Picture

Reachability
✓

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Generalized
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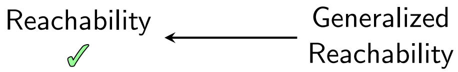

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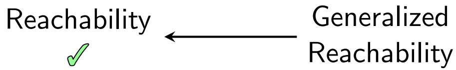
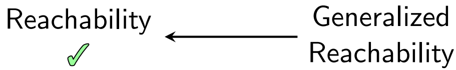


The Big Picture

Quantitative

Qualitative



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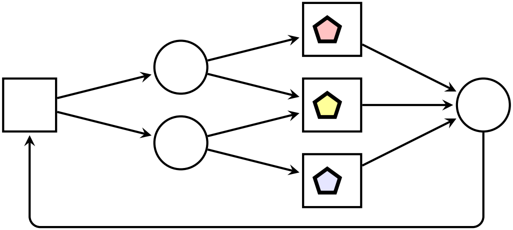

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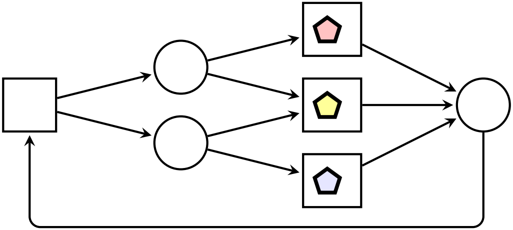
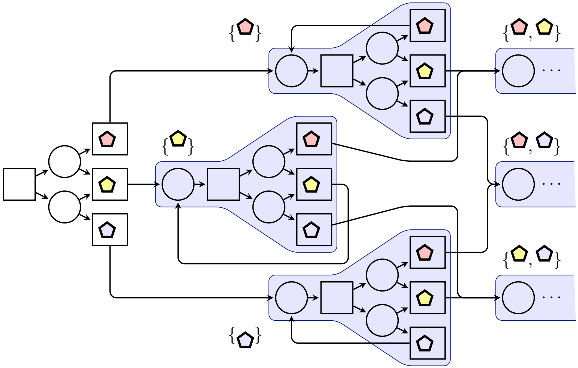


$$\mathrm{Cst}(\rho) = \begin{cases} 0 & \text{if } \{⬠, ⬠\} \text{ and } \{⬠, ⬠\} \text{ are visited} \\ 1 & \text{if one of them is visited} \\ 2 & \text{if neither is visited} \end{cases}$$

Quant. Gen. Reach.: Three Solutions



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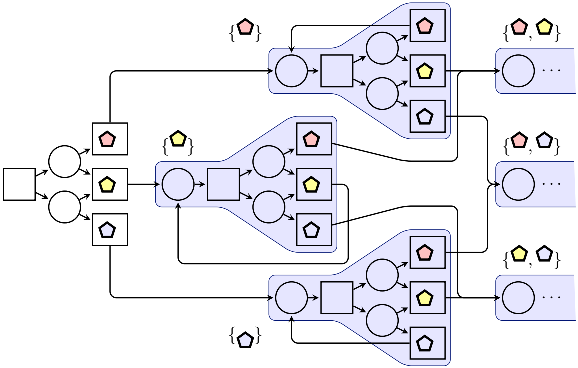

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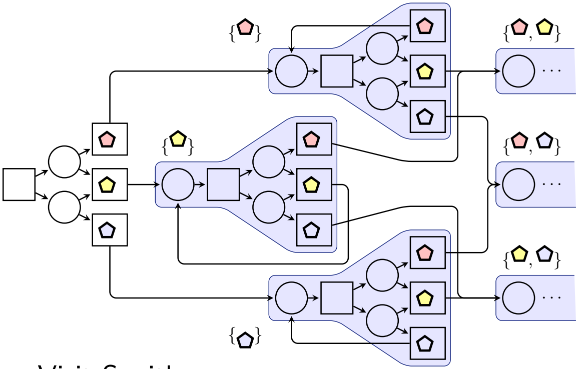

Win = Visit S with

$$\begin{cases} \{⬠, ⬠\} \subseteq S \text{ and } \{⬠, ⬠\} \subseteq S & \text{for } \mathrm{Cst} = 0 \\ & \\ & \end{cases}$$

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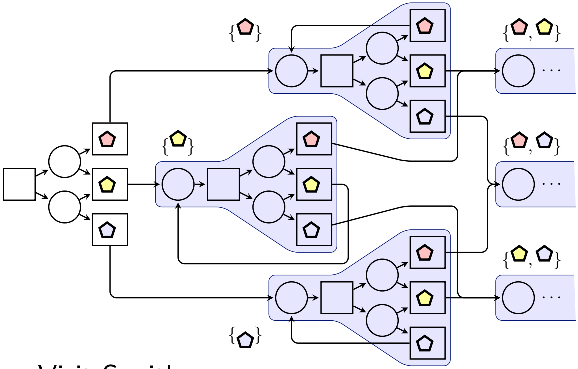

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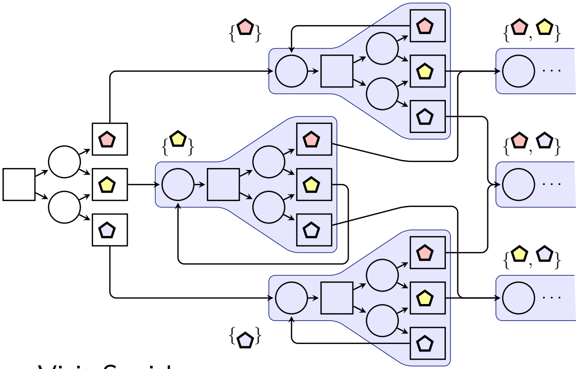

$Win =$ Visit S with

$$\begin{cases} \{⬠_{\text{red}}, ⬠_{\text{yellow}}\} \subseteq S \text{ and } \{⬠_{\text{yellow}}, ⬠_{\text{blue}}\} \subseteq S & \text{for } \mathrm{Cst} = 0 \\ \{⬠_{\text{red}}, ⬠_{\text{yellow}}\} \subseteq S \text{ or } \{⬠_{\text{yellow}}, ⬠_{\text{blue}}\} \subseteq S & \text{for } \mathrm{Cst} = 1 \\ S \text{ arbitrary} & \text{for } \mathrm{Cst} = 2 \end{cases}$$

The Big Picture

Quantitative Generalized Reachability

Quantitative

Qualitative

Reachability ✓ ← Generalized Reachability

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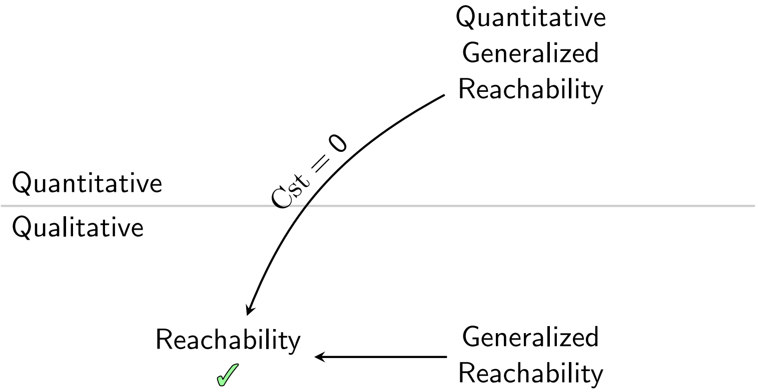

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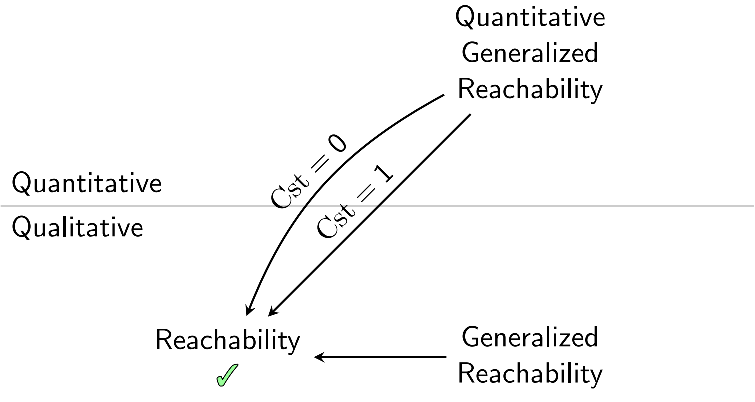

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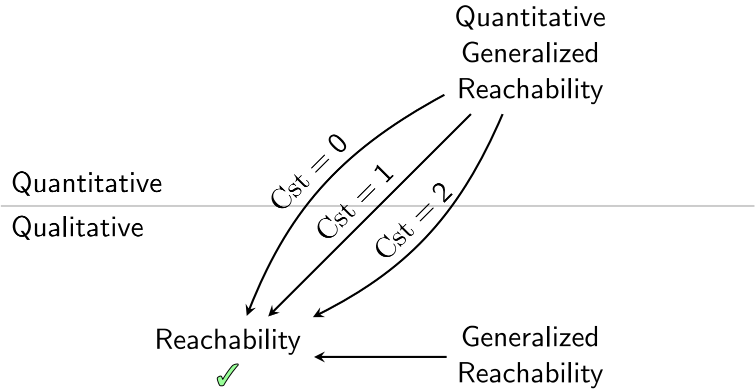

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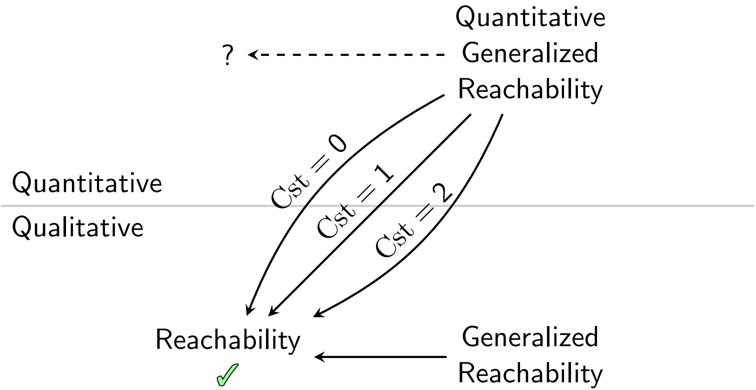

Quantitative Reductions

Let $\mathcal{G}$ and $\mathcal{G}'$ be games.

$\mathcal{G} \leq_M \mathcal{G}'$

iff

- $\mathcal{G}'$ is ($\mathcal{G}+$ memory structure) and
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Theorem

If Player 0 can win in $\mathcal{G}'$, then she can win in $\mathcal{G}$ using $\mathcal{M}$.

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Let $\mathcal{G}$ and $\mathcal{G}'$ be quantitative games.

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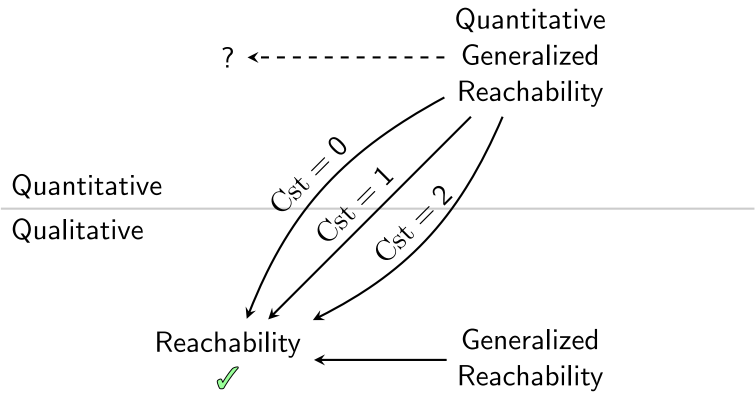
Theorem

If Player 0 can ~~win~~ enforce $\mathrm{Cst} \leq b$ in $\mathcal{G}'$, then she can ~~win~~ enforce $\mathrm{Cst} \leq b$ in $\mathcal{G}$ using $\mathcal{M}$.

The Big Picture



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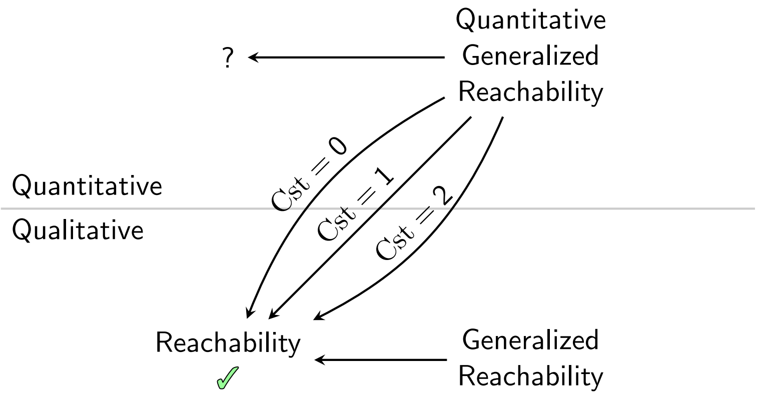
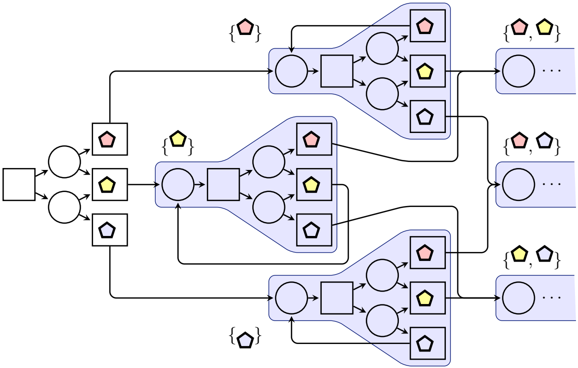


Recap: Quant. Gen. Reach.



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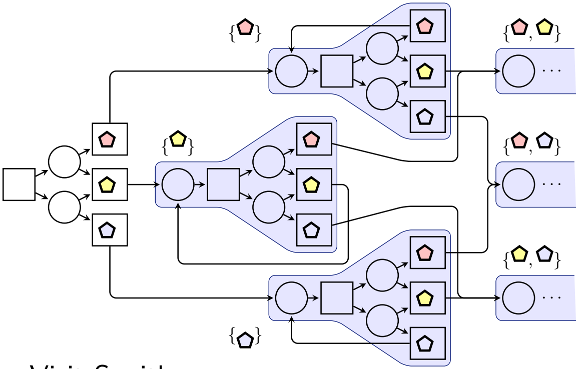

Win = Visit S with

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- S arbitrary for $\mathrm{Cst} = 2$

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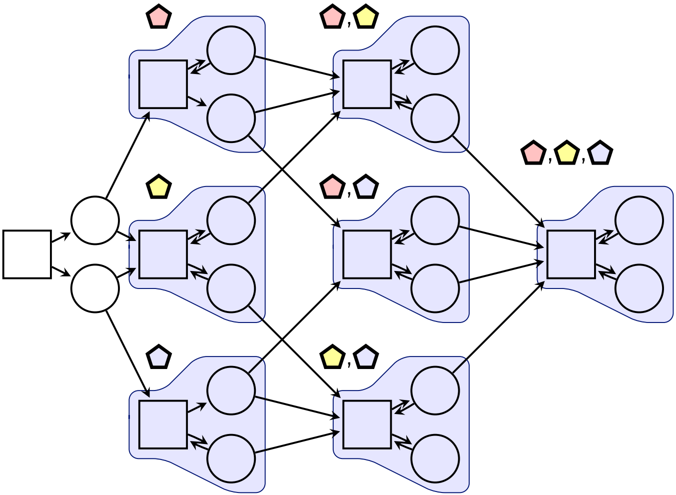

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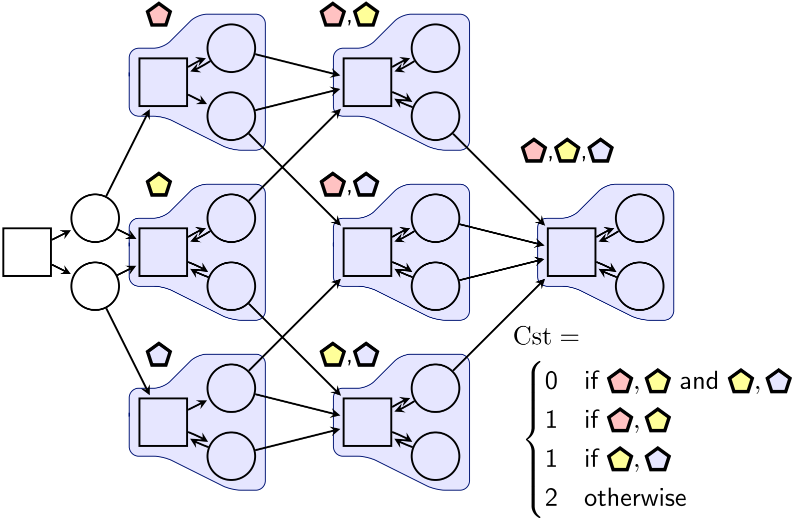

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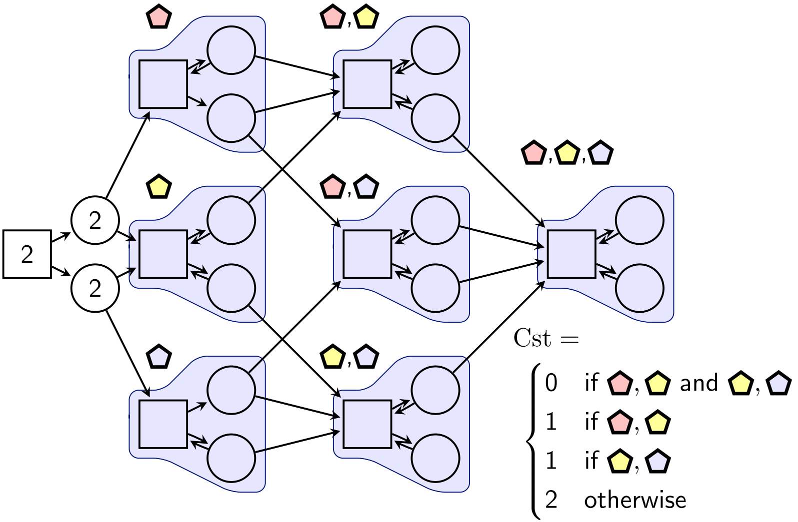

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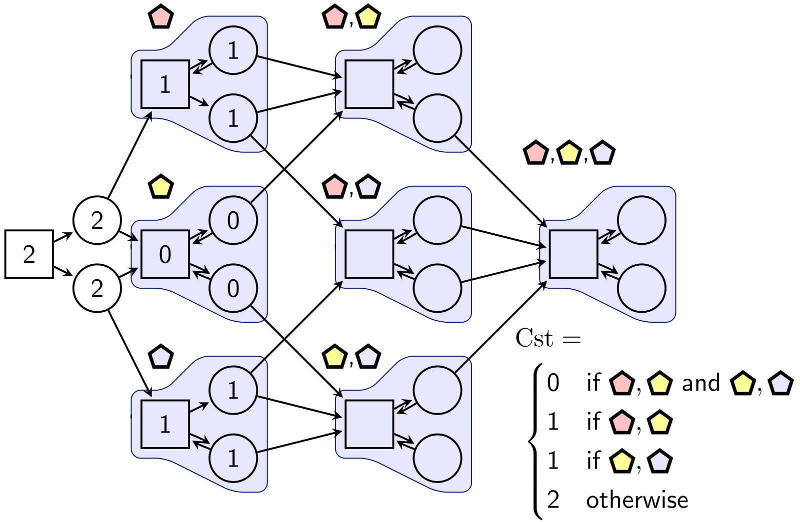

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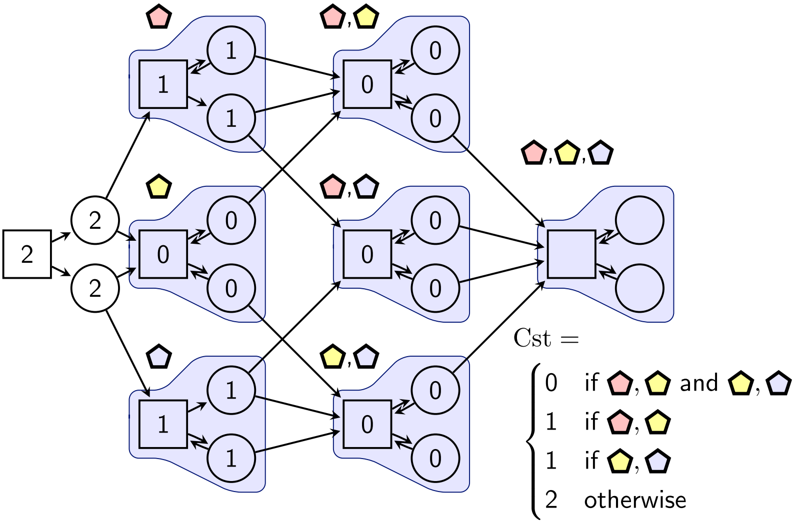

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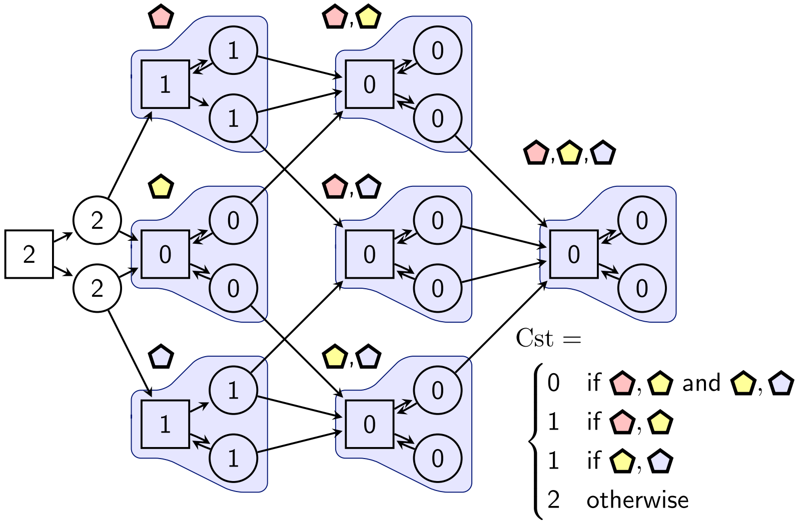

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Ingredients:

- Vertex set V,

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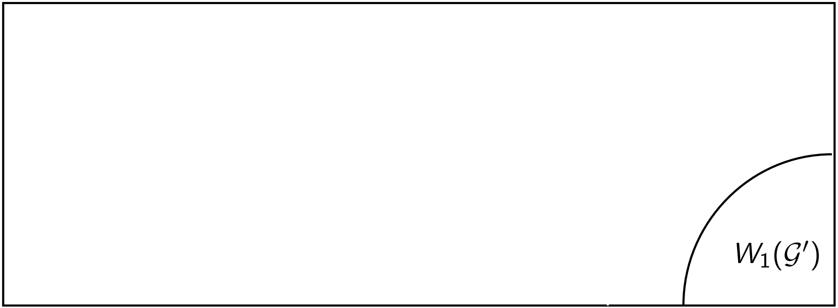

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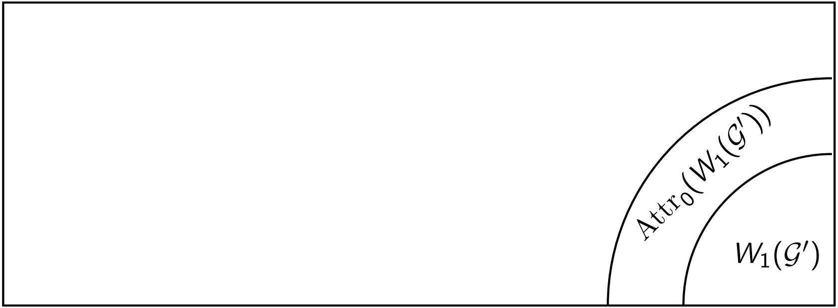

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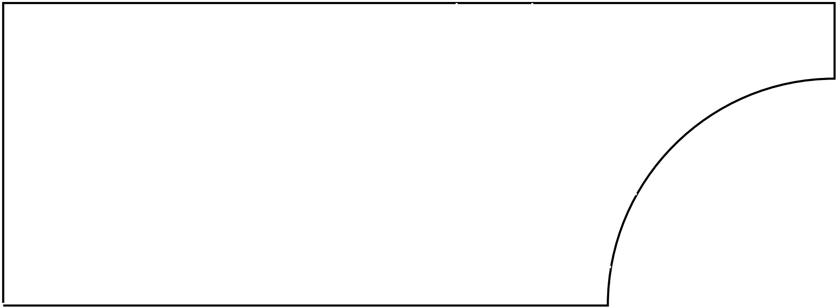

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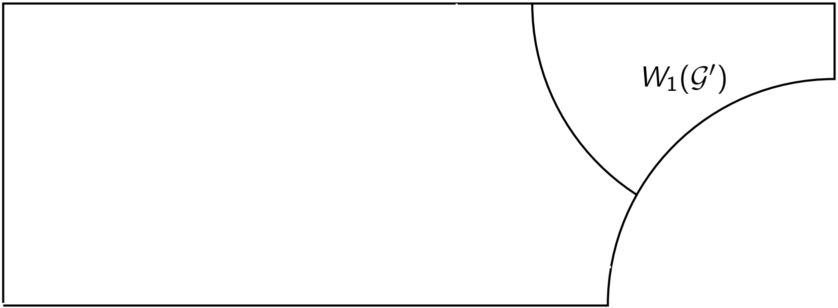
Given: Vertex-ranked lim sup-game $\mathcal{G}$, corresponding sup-game $\mathcal{G}'$.



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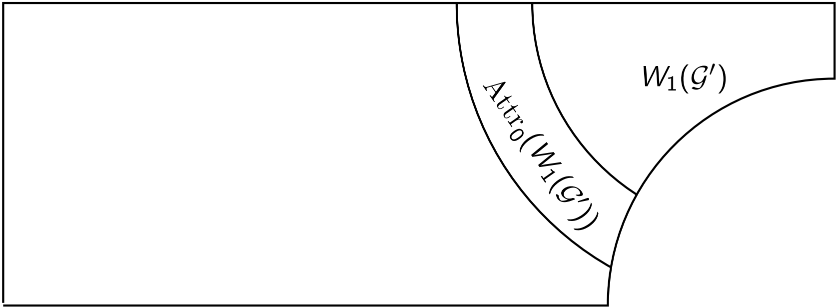

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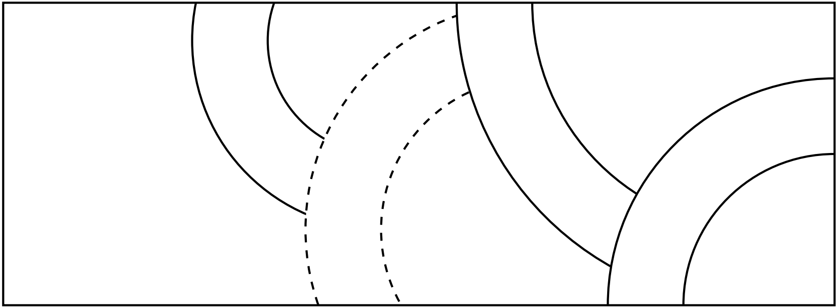

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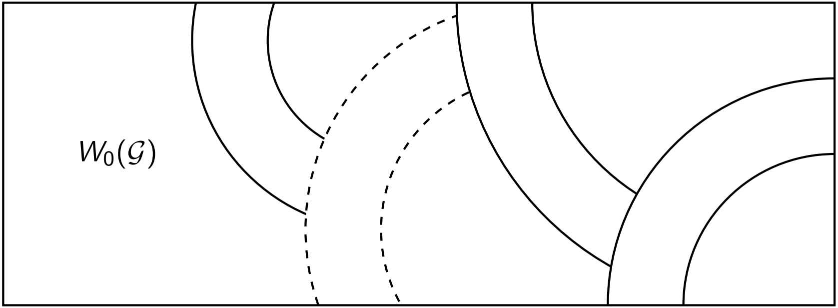

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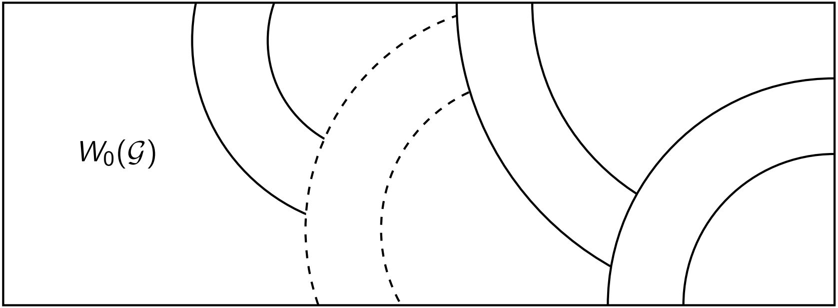

Theorem

Each vertex-ranked lim sup*-game can be solved by solving polynomially many* sup*-games with the same winning condition*

Solving Vertex-Ranked sup-Games

Given: Vertex-Ranked $\limsup$-game $\mathcal{G}$, vertex v, threshold b

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Solution Idea:

1. Find vertices of rank $> b$,

Solving Vertex-Ranked sup-Games

Given: Vertex-Ranked $\limsup$-game $\mathcal{G}$, vertex v, threshold b

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1. Find vertices of rank $> b$,
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Given: Vertex-Ranked $\limsup$-game $\mathcal{G}$, vertex v, threshold b

Question: Can Player 0 ensure cost at most b from v?

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1. Find vertices of rank $> b$,
2. remove 1-attractor of those vertices,
3. solve resulting game qualitatively.

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Given: Vertex-Ranked $\limsup$-game $\mathcal{G}$, vertex v, threshold b

Question: Can Player 0 ensure cost at most b from v?

Solution Idea:

1. Find vertices of rank $> b$,
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Theorem

Vertex-Ranked sup*-games can be solved with only an additive overhead over the qualitative solution.*

Putting it all Together

Theorem

Vertex-ranked $\limsup$*-game can be solved by solving polynomially many* $\sup$*-games with the same winning condition*

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Putting it all Together

Theorem

Vertex-ranked $\limsup$-*game can be solved by solving polynomially many* sup-*games with the same winning condition*

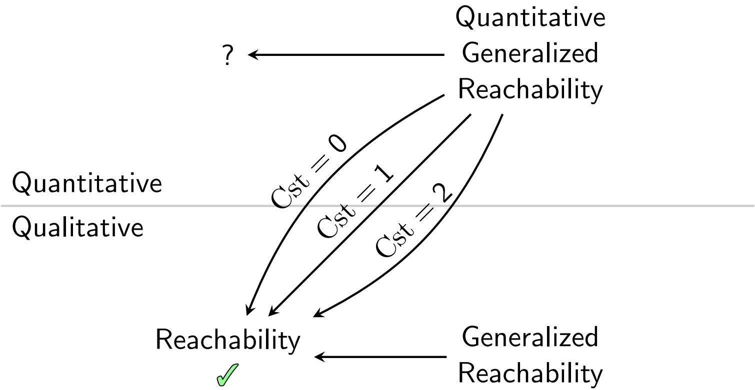
Theorem

Vertex-ranked sup-*games can be solved with only an additive overhead over the qualitative solution.*

Corollary

Vertex-ranked $\limsup$-*game can be solved with only a polynomial overhead*

The Big Picture



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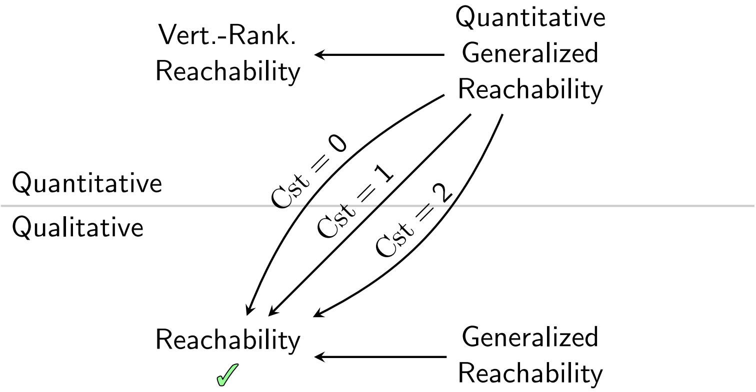

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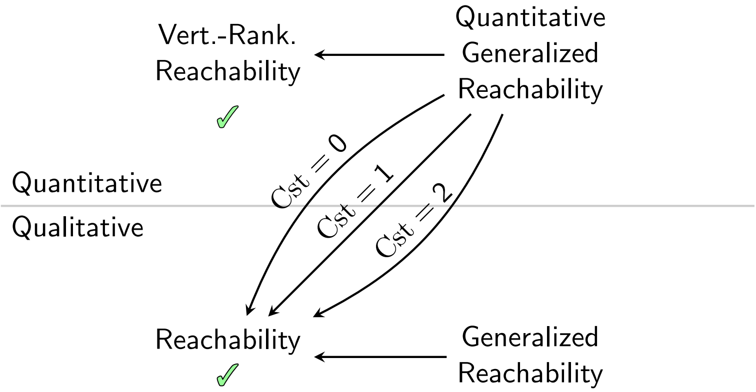

In the Paper

- Strategies witnessing results for vertex-ranked games

In the Paper

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- Quantitative Reductions for Request-Response games

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- Strategies witnessing results for vertex-ranked games
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- Fault resilient strategies

Conclusion

Contribution

- Lifted reductions to quantitative games

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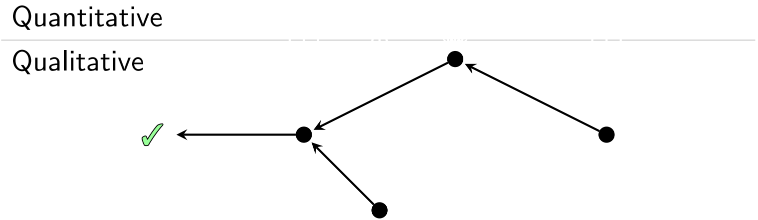
Next Steps



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Contribution

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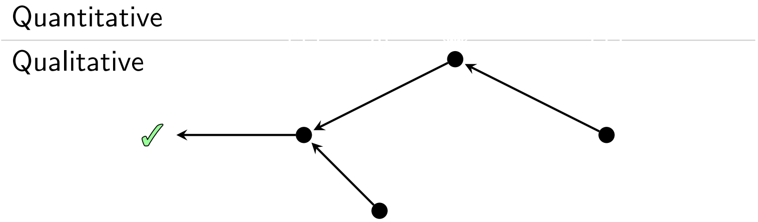

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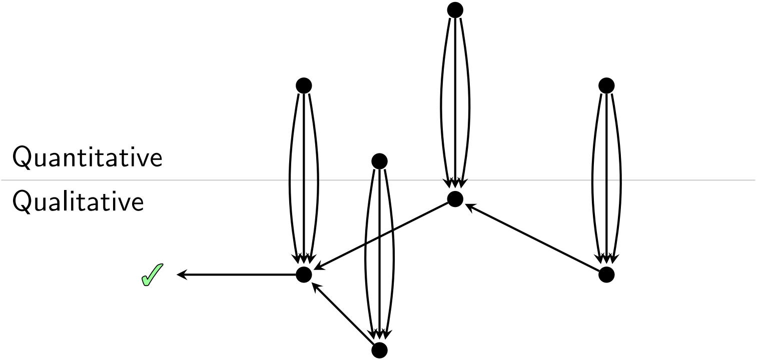

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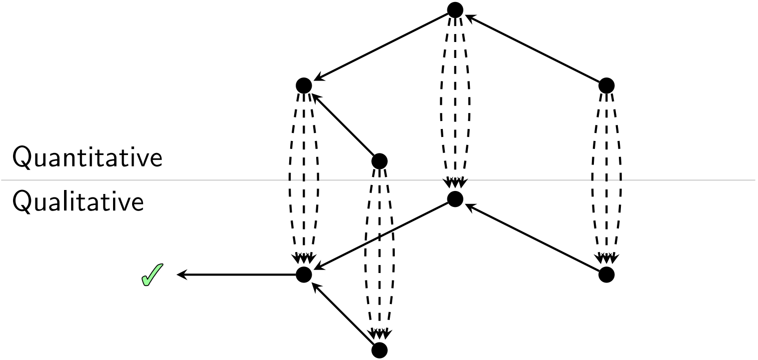

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Next Steps